THE REPUGNANT CONCLUSION AND UTILITARIANISM UNDER DOMAIN RESTRICTIONS

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Abstract

The paper investigates the problem of the repugnant conclusion in variable-population social choice. A choice rule exhibits the repugnant conclusion if, given any alternative in which all individuals enjoy a high quality of life, there is always a better alternative (according to the choice rule) in which the population is larger and each individual has a life which is barely worth living. While it is well-known that classical utilitarianism (CU) exhibits the repugnant conclusion on an unrestricted domain, the paper shows that critical-level utilitarianism (CLU) also exhibits the repugnant conclusion on an unrestricted domain when repugnance is defined in terms of the critical level. In contrast, both CU and CLU are shown to avoid the repugnant conclusion on a restricted domain defined by bounded resources, the law of conservation of matter, concavity and monotonicity of preferences, positive subsistence consumption, and positive “neutral consumption.” In light of the universality of these restrictions, one may conclude that the problem of the repugnant conclusion may not be as great as previously thought.

1. Introduction

Avoidance of the repugnant conclusion remains a central preoccupation of variable-population social choice theory. As defined by Parfit (1984), a social choice rule exhibits the repugnant conclusion if, given any population in which all members enjoy a high quality of life, there is always a better
alternative (according to the choice rule) in which the population is larger and each member has a life which is barely worth living. Parfit shows that classical utilitarianism (CU), defined as the sum of individual utility levels, exhibits the repugnant conclusion.

In response, Blackorby and Donaldson (1984) and Blackorby, Bossert, and Donaldson (1995) have proposed critical-level utilitarianism (CLU). A CLU social choice rule is defined as the sum of deviations of individual utility from a given value referred to as the critical level. It follows that the addition to an existing population of an individual with utility at the critical level, with no other changes, leaves the social planner indifferent between adding and not adding the individual.

The critical level has its analogue for individuals in the concept of neutrality. Neutrality is the utility level at which an individual is indifferent between her actual life and a hypothetical life devoid of experiences. It is conventional to fix the neutral level at a value of zero. If the critical level is greater than neutrality, then the CLU social choice rule avoids the repugnant conclusion.

Unfortunately, as noted by Blackorby, Bossert, and Donaldson (1997, 2002a,b, 2003), having the critical level greater than neutrality is incompatible with the Pareto-plus principle of Sikora (1978). The Pareto-plus principle states that the addition of any individual with utility above neutrality (or following convention, with utility greater than zero) to an existing population with no other changes results in an increase in social welfare. This principle has some appeal, and therefore one may believe that a good social choice rule should satisfy Pareto-plus. CU does satisfy it, and therefore one would appear to be faced with a conundrum: either embrace CU which satisfies Pareto-plus but exhibits the repugnant conclusion, or embrace CLU (with critical level greater than neutrality) which avoids the repugnant conclusion but violates Pareto-plus.

This difficulty extends beyond CU and CLU. Blackorby, Bossert, and Donaldson (2003) propose a list of reasonable properties and show that there is no welfarist social choice rule which satisfies all of these properties. Ng (1989) and Arrhenius (2000) report similarly negative results.

The present paper provides a new perspective on this conundrum. In particular, it will be argued that the adoption of a critical level greater than neutrality under CLU simply exchanges one repugnant conclusion, based on neutrality, for another, based on the critical level. In this view, the escape from the repugnant conclusion offered by the CLU family is not entirely satisfying. Therefore, in the comparison of CU and CLU, one may be inclined toward CU, since at least it satisfies Pareto-plus.

\[^{1}\text{For the record, these properties are: strong Pareto, continuity, anonymity, existence independence, existence of critical levels, non-negative critical levels, avoidance of the repugnant conclusion, and avoidance of the strong sadistic conclusion. See Blackorby et al. (2003) for details. A social choice rule is welfarist if it takes into account individuals' utility levels only and disregards how the utility levels are achieved.}\]
Further, it seems reasonable to change approach in considering the repugnant conclusion. It is conventional, in assessing the suitability of social choice rules, to consider their performance on an unrestricted domain of feasible states. In contrast, economic models typically impose a significant amount of structure on preferences and technological possibilities. Given the impossibility results noted above, it may be profitable to inquire what restrictions on preferences and technology would be sufficient in order for certain rules to avoid the repugnant conclusion. This approach has generated a large literature in the context of Arrow’s impossibility theorem (see LeBreton and Weymark 2002), but it has not yet been applied to the problem of the repugnant conclusion. At the very least, the strength or weakness of these restrictions could provide some insight into how serious the problem is.

Blackorby et al. (2002b and 2003) show that, of 13 welfarist social choice rules examined, only classical generalized utilitarianism and critical-level generalized utilitarianism satisfy a truncated version of their list of desirable properties. Therefore, it seems reasonable to focus attention on these two rules. At the same time, Blackorby and Donaldson (1982) demonstrate the severe restrictions on the generalized form which result from the need to accommodate negative utilities in an informational environment where the origin is socially meaningful. For this reason, the present paper focuses on the non-generalized forms of these rules, i.e., CU and CLU.

The value of the inquiry into domain restrictions depends upon the restrictions being reasonably weak. For this reason, the paper focuses on some basic properties of physics and preferences, in particular (i) bounded resources, (ii) the law of conservation of matter, (iii) concavity of preferences, (iv) monotonicity of preferences, (v) positive subsistence consumption, and (vi) positive neutral consumption. It turns out that, taken together, these restrictions are sufficient for avoiding the repugnant conclusion under CU and CLU. In light of the universality of these properties, it appears that the problem of the repugnant conclusion may not be as serious as was thought.

In addition, it is shown that CU and CLU avoid the repugnant conclusion under conditions (i)–(iv), (vi), and (vii) subsistence consumption equal to zero, provided (viii) individuals have identical preferences. The assumption of identical preferences is a strong one. Nonetheless, the result has technical interest, as economic models frequently make this assumption.

The paper is structured as follows. Section II introduces notation and definitions. Section III presents the results without domain restrictions. Section IV presents the results under domain restrictions. Section V concludes.

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2 The truncated list includes: strong Pareto, continuity, anonymity, existence independence, existence of critical levels, and non-negative critical levels.

3 As defined below, neutral consumption is the level of material consumption corresponding with neutral utility.
2. Notation and Definitions

2.1. Consumption and Individual Preferences

Let $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^{++}$ denote the set of real numbers, non-negative real numbers, and positive real numbers, respectively. Let $\mathbb{Z}^+$ denote the set of positive integers.

Define $N \in \mathbb{Z}^+$ as population size and $i \in \{1, \ldots, N\}$ as an individual in the population. There are $m \in \mathbb{Z}^+$ consumption goods, indexed by $j \in \{1, \ldots, m\}$. The consumption of good $j$ by individual $i$ is denoted $x_{ij}$, and the individual’s consumption vector is $x_i = (x_{i1}, x_{i2}, \ldots, x_{im})$. The set of admissible consumption vectors is denoted $X \subseteq \mathbb{R}^m_+$. An allocation $x$ consists of the consumption vectors of all individuals in the population; i.e., $x = \{x_i\}_{i \in \{1, \ldots, N\}}$. The set of admissible allocations of population size $N$ is denoted $X^N$ (the $n$-fold Cartesian product of $X$).

An individual’s lifetime utility is given by the function $U_i : X \to \mathbb{R}^1$. It is convenient though not essential to assume differentiability. The set of all possible utility functions is denoted $\xi$, and $U = \{U_i\}_{i \in \{1, \ldots, N\}}$ represents a profile of utility functions where each individual may have a different function. An individual’s utility level is given by $u_i = U_i(x_i)$. Given an allocation $x$ and profile $U$, a realization is defined as the set of utility levels, $U(x) = \{U_i(x_i)\}_{i \in \{1, \ldots, N\}} = \{u_i\}_{i \in \{1, \ldots, N\}} = u$.

For the purpose of making judgements regarding the addition of new individuals, variable-population social choice requires the definition of a neutral value of utility. Blackorby, Bossert, and Donaldson (2002a) interpret neutrality as the level of lifetime utility at which an individual is indifferent between her actual life and a hypothetical life devoid of experiences. The convention is followed here that a value of zero represents neutrality.

The focus on lifetime measures of utility means that time does not enter explicitly into the analysis; rather, each alternative allocation $x$ represents a complete history of the world, which is to be ranked. A comprehensive notion of utility is intended, in which well-being is derived from both material and non-material goods, including such things as health, longevity, family, friendships, career, education, recreation, or participation in the community.

A material good is derived from a resource endowment, $E \in \mathbb{R}^{++}$, and labor input equal to population, $N$, by means of a production technology. For simplicity, a single material good is assumed, identified as good 1. Its technology is represented by the non-decreasing supply function $S(N, E)$. The remaining goods, indexed by $j = 2, \ldots, m$, are considered non-material. An upper bound on resources is denoted $\bar{E}$.

Given the resource endowment, the law of conservation of matter entails an upper-bound on the supply of the material good, denoted $\bar{S}(E)$. It follows
that the marginal product of labor (first difference for integer-valued $N$) and the average product of labor must converge to 0.

Subsistence consumption, $c_s \in \mathbb{R}_+$, is the minimum level of the material good required to sustain life (assumed identical for all individuals). $\mathbb{R}_+^1$ represents the set of real values not lower than $c_s$.

A particular form of monotonicity of preferences is used later, according to which utility is strictly increasing in the material good and weakly increasing in non-material goods.

**DEFINITION 1 (Conservation of Matter):** For any $E \in \mathbb{R}_{++}$, (i) there exists a value $\bar{S}(E) \in \mathbb{R}_{++}$ such that $S(N, E) \leq \bar{S}(E)$ for all $N \in \mathbb{Z}_{++}$, and (ii) $\lim_{N \to \infty} [S(N + 1, E) - S(N, E)] = \lim_{N \to \infty} S(N, E)/N = 0$.

**DEFINITION 2 (Monotonicity):** For any utility function $U_i \in \xi$ and consumption vector $x_i \in X$, $\partial U_i(x_i)/\partial x_{i1} > 0$ and $\partial U_i(x_i)/\partial x_{i1} \geq 0$ for $j \in \{2, \ldots, m\}$.

Neutral consumption is the level of the material good which corresponds with the neutral utility level (zero by convention).

**DEFINITION 3 (Neutral Consumption):** For individual $i \in \mathbb{Z}_{++}$, with utility function $U_i \in \xi$, and non-material goods $(x_{i2}, \ldots, x_{im}) \in \mathbb{R}^{m-1}_{++}$, neutral consumption $c_{n,i} \in \mathbb{R}_+$ is defined by the relationship $U_i(c_{n,i}, x_{i2}, \ldots, x_{im}) = 0$.

In the absence of restrictions on substitutability, this relationship implies a function $c_{n,i} = C_{n,i}(x_{i2}, \ldots, x_{im})$, which, by monotonicity, is non-increasing in its arguments. Neutral consumption cannot be less than subsistence consumption by definition, although it can be undefined, as when utility is strictly positive for any $x_{i1} \geq c_s$. Yet, it seems more plausible that $c_{n,i}$ should be defined, since low levels of consumption bring hunger, disease, indignity, and pain. The following restriction is considered in the sequel.

**DEFINITION 4 (Lower-bound Neutral Consumption):** Given $i \in \mathbb{Z}_{++}$, $U_i \in \xi$ and $c_s \in \mathbb{R}_+$, there exists a value $\bar{c}_{n,i} \geq c_s$ such that $C_{n,i}(x_{i2}, \ldots, x_{im}) \geq c_{n,i}$ for all $(x_{i2}, \ldots, x_{im}) \in \mathbb{R}^{m-1}_{++}$.

This restriction also entails a common lower bound.

**DEFINITION 5 (Common Lower-bound Neutral Consumption):** Given $c_s \in \mathbb{R}_+$, there exists a value $\bar{c}_{n} \geq c_s$ such that $c_{n,i} \geq \bar{c}_{n}$ for all $i \in \mathbb{Z}_{++}$ and $U_i \in \xi$.

**2.2. Concavity and Bounded Utility**

The class of concave utility functions is denoted $\xi_c \subset \xi$. For variable-population comparisons, utilitarianism requires that individual utility be invariant up to a common, positive scaling factor (ratio-scale full comparability
for CU and reference ratio-scale full comparability for CLU). Since concavity is preserved under such transformations, it is a meaningful property in the present context.

The combination of concavity and lower-bound neutral consumption implies a concave upper-bound for each utility function, as follows.

**LEMMA 1 (Concave upper-bound utility):** Given $c_s \in \mathbb{R}^+$ and $i \in Z^{++}$, for every concave utility function $U_i \in \xi_c$ with a lower-bound value of neutral consumption, $\xi_{n,i} \geq c_s$, there exists a concave function $\overline{U}_i : \mathbb{R}^1_{\geq c_s} \rightarrow \mathbb{R}^1$ such that, for any $x_i \in \mathbb{R}^1_{\geq c_s}$, $\overline{U}_i(x_{i1}, x_{i2}, \ldots, x_{im}) \leq \overline{U}_i(x_{i1})$ for all $(x_{i2}, \ldots, x_{im}) \in \mathbb{R}^{m-1}$ and $\lim_{x_{i2}, \ldots, x_{im} \to \infty} \overline{U}_i(x_{i1}, x_{i2}, \ldots, x_{im}) = \overline{U}_i(x_{i1})$. Further, monotonicity entails $U_i(c_{\bar{n}}, i) = 0$.

**Proof:** Consider any two allocations of goods $x_i = (x_{i1}, x_{i2}, \ldots, x_{im}) \in X$ and $x'_i = (x'_{i1}, x_{i2}, \ldots, x_{im}) \in X$ such that $x_{i1} \leq c_{\bar{n}}, i < x'_{i1}$ and which are identical in the non-material goods $x_{i2}, \ldots, x_{im}$. By the intermediacy of $c_{\bar{n}}, i$, there exists a scalar $\lambda \in [0, 1]$ such that

$$\lambda x_{i1} + (1 - \lambda) x'_{i1} = c_{\bar{n}, i}. \tag{1}$$

By concavity

$$\lambda U_i(x_i) + (1 - \lambda) U_i(x'_i) \leq U_i(\lambda x_i + (1 - \lambda) x'_i). \tag{2}$$

Furthermore, $U_i(\lambda x_i + (1 - \lambda) x'_i) = U_i(c_{\bar{n}, i}, x_{i2}, \ldots, x_{im})$ by (1), and by definition of the lower-bound neutral value it follows that $U_i(c_{\bar{n}, i}, x_{i2}, \ldots, x_{im}) \leq U_i(C_{n,i}(x_{i2}, \ldots, x_{im}), x_{i2}, \ldots, x_{im}) = 0$. Therefore (2) becomes

$$\lambda U_i(x_i) + (1 - \lambda) U_i(x'_i) \leq 0. \tag{3}$$

Assume by way of contradiction that there is no upper-bound on the utility function. In this case, $U_i(x_i)$ and $U_i(x'_i)$ can be made arbitrarily large by increasing the allocation of non-material goods arbitrarily. But then (3) is violated. Therefore an upper bound function must exist.

By concavity, (2) holds for any two allocations $x_i, x'_i \in X$. It follows that, given any two values $x_{i1}, x'_{i1}$ of the material good,

$$\lim_{x_{i2}, \ldots, x_{im} \to \infty} [\lambda U_i(x_i) + (1 - \lambda) U_i(x'_i)] \leq \lim_{x_{i2}, \ldots, x_{im} \to \infty} U_i(\lambda x_i + (1 - \lambda) x'_i),$$

or equivalently,

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5Blackorby, Bossert, and Donaldson (1999).
\( \lambda U_i(x_{i1}) + (1 - \lambda) U_i(x'_{i1}) \leq U_i(\lambda x_{i1} + (1 - \lambda) x'_{i1}) \).

Therefore, \( \bar{U}_i \) is concave.

It follows from the definitions of lower-bound neutrality and monotonicity that \( \bar{U}_i(\xi_{n,i}) = 0 \); i.e., \( \xi_{n,i} \) is associated with the upper-bound utility function \( \bar{U}_i \). ■

The existence of an upper-bound utility function, \( \bar{U}_i(x_{i1}) \), entails a form of diminishing marginal substitutability between material and non-material goods. In particular, under this property, one cannot make utility arbitrarily large simply by adding more non-material goods without also adding more material goods, since in the limit material and non-material goods become strict complements.

The subset of upper-bounded concave utility functions is denoted \( \bar{\xi}_c \).

Among other properties, the \( \bar{U}_i \) functions are differentiated by their neutral consumption values, \( \xi_{n,i} \), and their slopes at \( \xi_{n,i} \). For any two scalar values \( 0 < b < \bar{b} \), define \( \bar{\xi}_c^{b,b} \subset \bar{\xi}_c \) as the subset of upper-bounded concave utility functions \( U_i \) with bounds \( \bar{U}_i \) such that \( b \leq \frac{\partial U_i(\xi_{n,i})}{\partial x_{i1}} \leq \bar{b} \); i.e., with slopes at \( \xi_{n,i} \) bounded by \( b \) and \( \bar{b} \). Further, let \( \bar{\xi}_c^{b,b} \subset \bar{\xi}_c^{b,b} \) represent the subset of such functions for which \( \xi_{n,i} = \bar{\xi}_c \).

The class of quasi-linear utility functions plays a special role in the subsequent analysis. Let \( \xi_\ell \) represent the class of such functions, defined by

\[
U_i(x_i) = a + b(x_{i1} - c_\ell) + \psi(x_{i2}, \ldots, x_{im}),
\]

with \( a \in \mathbb{R} \), \( b \in \mathbb{R}_{++} \), and \( \psi : \mathbb{R}_{++}^{m-1} \to \mathbb{R}_+ \). Neutral consumption is defined by the relation

\[
a + b(c_{n,i} - c_\ell) + \psi(x_{i2}, \ldots, x_{im}) = 0,
\]

which, upon rearranging, indicates

\[
a = -b(c_{n,i} - c_\ell) - \psi(x_{i2}, \ldots, x_{im}) \leq 0.
\]

The inequality follows from the non-negativity of \( (c_{n,i} - c_\ell) \) and \( \psi \).

The effect of a lower neutral bound is to place an upper bound on \( \psi \), as seen by substituting \( \xi_{n,i} \) into (5) for given values of \( a \), \( b \), and \( c_\ell \). Denote this bound \( \tilde{\psi}(a, b, c_\ell, \xi_{n,i}) \). Substitution of this value into (4) yields the upper-bound utility function \( \bar{U}_i(x_{i1}) \) for the particular values of \( a \), \( b \), \( c_\ell \), and \( \xi_{n,i} \). Figure 1 shows a graph of a quasi-linear utility function, with increasing values of \( \psi(\cdot) \) reflected in increasing values of the vertical intercept (with the origin at \( (c_\ell, 0) \)). The upper bound \( \bar{U}_i(x_{i1}) \) intersects the horizontal axis at \( \xi_{n,i} \) (assumed strictly greater than \( c_\ell \) in the figure).

An alternative formulation of the quasi-linear function is obtained by exploiting the dependence of \( c_{n,i} \) on the non-material allocation \( (x_{i2}, \ldots, x_{im}) \) through \( \psi(\cdot) \). Substituting for \( a \) from (5) into (4) yields the formulation

\[
U_i(x_i) = b(x_{i1} - c_{n,i}),
\]
which shows the vertical intercept in terms of $c_{n,i}$ instead of $\psi(\cdot)$. The upper bound in this formulation is

$$U_i(x_{i1}) = b(x_{i1} - c_{n,i}).$$  \hfill (7)

Let $\tilde{\xi}_\ell \subset \xi_\ell$ represent the class of upper-bound quasi-linear functions (7).

Quasi-linear utility functions are differentiated by their slopes $b$ and neutral consumption level $c_{n,i}$ as shown in (6). As above, consider any two values $0 < b < \hat{b}$, and define the subset $\xi^{b,\hat{b}}_\ell \subset \xi_\ell$ as the class of all quasi-linear utility functions with minimum slope $\hat{b}$ and maximum slope $\hat{b}$; i.e., for any $U_i \in \xi^{b,\hat{b}}_\ell$, $b \leq b \leq \hat{b}$. Further define the subset $\tilde{\xi}^{b,\hat{b}}_\ell \subset (\xi^{b,\hat{b}}_\ell \cap \tilde{\xi}_\ell)$ as the class of all upper-bound quasi-linear functions $\bar{U}_i \in \tilde{\xi}_\ell$ with slopes between $b$ and $\hat{b}$. As observed, these functions correspond with lower-bound neutral values $c_{n,i}$. Finally, define the subset $\tilde{\xi}^{b,\hat{b}}_{\ell,b} \subset \tilde{\xi}^{b,\hat{b}}_\ell$ as the class of all upper-bound quasi-linear functions with slopes bounded by $b$ and $\hat{b}$, and with $c_{n,i}$ equal to the common lower-bound value $\bar{c}_{n}$. 

## 2.3. Social Preferences

A social alternative consists of a population size $N$ and a corresponding allocation $x \in X^N$. The set of non-empty social alternatives is $\bigcup_{N \in \mathbb{Z}_+} X^N$. 
Social preferences are expressed by an ordering, $R$, over the set of social alternatives, where an ordering is defined as a complete, reflexive and transitive set of binary relations. An ordering is welfarist if the relevant information about the alternative is summarized by the utility realization $u = (u_1, \ldots, u_N)$. For $N, N' \in \mathbb{Z}^+$ and $u \in \mathbb{R}^N$ and $u' \in \mathbb{R}^{N'}$, the statement $uRu'$ is translated as “$u$ is at least as good as $u'$.” The ordering has symmetric and asymmetric parts, $I$ and $P$, corresponding with indifference and strict preference respectively, which are defined in the usual manner.

The following properties of orderings are invoked below: continuity, anonymity, and weak Pareto.

**Definition 6 (Continuity):** For all $N \in \mathbb{Z}^+$ and all $u \in \mathbb{R}^N$, the sets \{ $u' \in \mathbb{R}^N \mid uRu'$ \} and \{ $u' \in \mathbb{R}^N \mid uRu'$ \} are closed.

**Definition 7 (Anonymity):** For all $N \in \mathbb{Z}^+$, all $u \in \mathbb{R}^N$, and all bijective mappings $b : \{1, \ldots, N\} \rightarrow \{1, \ldots, N\}$, $uI(u_{b(1)}, \ldots, u_{b(N)})$.

**Definition 8 (Weak Pareto):** For all $N \in \mathbb{Z}^+$ and all $u, u' \in \mathbb{R}^N$, if $u_i > u'_i$ for all $i \in \{1, \ldots, N\}$, then $uPu'$.

Comparison of variable-population social choice rules is facilitated by the following representation theorem, due to Blackorby and Donaldson (1984). Define representative utility as a value $\nu(u) \in [\min \{u\}, \max \{u\}]$ such that $u$ is socially indifferent to an alternative with the same population in which every member receives utility of $\nu$.

**Representation Theorem:** For every welfarist variable-population ordering $R$ that satisfies continuity, anonymity, and weak Pareto, there exists a continuous function, $W : \mathbb{Z}^+ \times \mathbb{R} \rightarrow \mathbb{R}$, which represents the ordering in terms of population size and representative utility only; i.e., for all $N, N' \in \mathbb{Z}^+$ and $u \in \mathbb{R}^N$, $u' \in \mathbb{R}^{N'}$,

$$uRu' \Leftrightarrow W(N, \nu(u)) \geq W(N', \nu(u')).$$  

(8)

Comparisons can also be made with the null alternative, denoted $\emptyset$, which is defined as the alternative in which no one exists. The corresponding value of $W$ for the null alternative is zero. Thus, for $N \in \mathbb{Z}^+$ and $u \in \mathbb{R}^N$, (8) becomes

$$uR\emptyset \Leftrightarrow W(N, \nu(u)) \geq 0.$$  

Blackorby et al. (1984, 1995) define the critical level $\gamma$ as a value of lifetime utility such that any given alternative (including the null alternative) is socially indifferent to an alternative which is identical except that it includes an extra individual with utility equal to $\gamma$.

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6The presentation here follows Blackorby, Bossert, and Donaldson (2002a).
DEFINITION 9 (Critical Level): A value $\gamma \in \mathbb{R}$ is a critical level of the ordering $R$ if, for any $N \in \mathbb{Z}_{++}$ and either $u \in \mathbb{R}^N$ or $u = \emptyset$, $uI(u, \gamma)$.

It follows that an alternative of any population size $N \in \mathbb{Z}_{++}$ in which all individuals receive utility of $\gamma$ is socially indifferent to the null alternative. Such an alternative will be referred to as a null-equivalent.

CU ranks alternatives by the value of total utility. In particular, for $N, N' \in \mathbb{Z}_{++}$, $u \in \mathbb{R}^N$, and $u' \in \mathbb{R}^{N'}$

$$uRu' \iff \sum_{i=1}^{N} u_i \geq \sum_{i=1}^{N'} u'_i.$$ 

CLU ranks alternatives by the sum of deviations from the critical level, i.e.

$$uRu' \iff \sum_{i=1}^{N} (u_i - \gamma) \geq \sum_{i=1}^{N'} (u'_i - \gamma).$$

Thus, CLU establishes $\gamma$ as a threshold which new individuals must surpass in order to represent an improvement in social welfare.

Following the representation theorem, these evaluation functions can be rewritten in terms of population size and representative utility only. Under both CU and CLU, representative utility is equal to average utility, and thus the respective evaluation functions can be expressed as

$$W_{CU}(N, \upsilon) = N\upsilon$$ and $$W_{CLU}(N, \upsilon) = N[\upsilon - \gamma],$$

with domain $D \subseteq (\mathbb{Z}_{++} \times \mathbb{R})$. CU is obtained as a special case of CLU with $\gamma = 0$.

Figure 2 shows a plot of social indifference sets for CU and CLU. By construction, $W(N, \upsilon)$ is increasing in $N$ for $\upsilon > \gamma$ and constant at a value of zero for $\upsilon = \gamma$. The plot of each indifference set is asymptotic to the $\upsilon = \gamma$ axis, which reflects the trade-off between population and representative utility under utilitarianism.

3. The Repugnant Conclusion without Domain Restrictions

The following definition captures the notion of the repugnant conclusion based on neutrality.

DEFINITION 10 (Repugnant Conclusion (RC-0)): A variable-population welfare function $W(N, \upsilon)$ exhibits the repugnant conclusion if, for every $\alpha > 0$ and for every $(N, \upsilon) \in D$ with $\upsilon > 0$, there exists a value $\varepsilon \in (0, \alpha)$ and a population size $M > N$ such that $(M, \varepsilon) \in D$ and $W(M, \varepsilon) > W(N, \upsilon)$.

\footnote{For ease of identification, points belonging to the same indifference set have been connected.}
This definition states that, for any alternative with a given level of representative utility above neutrality, there is always another alternative with a higher population and representative utility arbitrarily close to neutrality which is socially preferred.

It follows immediately that CU exhibits RC-0, due to the asymptotic convergence of the social indifference plots. In particular, for any \( N \in \mathbb{Z}^+ \) and \( \nu > \varepsilon > 0 \), there exists an \( M \in \mathbb{Z}^+ \) such that \( M > Nu/\varepsilon \) and therefore \( W_{CU}(M, \varepsilon) = Me > Nu = W_{CU}(N, \nu) \).

In contrast, CLU avoids RC-0 provided \( \gamma > 0 \) (i.e., critical value greater than neutrality), since in that case the social indifference plots are asymptotic to the horizontal line \( \nu = \gamma > 0 \). In particular, for \( 0 < \varepsilon < \gamma \), \( W_{CLU}(M, \varepsilon) \) is negative and decreasing in \( M \). For this reason, Blackorby, Bossert, and Donaldson have argued for the use of CLU with positive critical level, and this assumption will be made for CLU in the sequel.

An alternative definition of the repugnant conclusion is based on the critical level, \( \gamma \).

**Definition 11 (Repugnant Conclusion (RC-\( \gamma \)))**: A variable-population welfare function \( W(N, \nu) \) exhibits the repugnant conclusion if, for every \( \alpha > 0 \) and for every \( (N, \nu) \in D \) with \( \nu > \gamma \), there exists a value \( \varepsilon \in (0, \alpha) \) and a population size \( M > N \) such that \( (M, \gamma + \varepsilon) \in D \) and \( W(M, \gamma + \varepsilon) > W(N, \nu) \).

This definition states that, for any alternative with a given level of representative utility above \( \gamma \), there is always another alternative with a higher population and representative utility arbitrarily close to \( \gamma \) which is socially preferred. Both CU and CLU exhibit RC-\( \gamma \), since in both cases the social indifference plots are asymptotic to the horizontal line \( \nu = \gamma \).
RC-γ would appear to be the more compelling definition of repugnance. To see this point, observe that a null equivalent (a social alternative in which everyone receives utility equal to γ) is not worth having, since it is socially indifferent to the null alternative, ∅, in which no one exists. Now, given any alternative (N, υ), RC-γ entails that there is always a better alternative arbitrarily close to a null-equivalent. But by virtue of closeness to a null-equivalent, this better alternative must be scarcely worth having. It follows that CLU does not provide a satisfactory escape from the repugnant conclusion.

4. The Repugnant Conclusion under Domain Restrictions

The following result provides necessary and sufficient conditions under which CU and CLU avoid the repugnant conclusion based on the critical level (RC-γ). These conditions involve a domain restriction on the CU and CLU evaluation functions such that, for population in excess of a certain value, representative utility does not exceed an upper bound. This upper bound is expressed as a diminishing function of population, φ(N), and is contiguous with a particular indifference set; i.e., the welfare value is constant along the upper bound, for all values of population.

THEOREM 1: CU and CLU avoid RC-γ if and only if there exist values  \( \hat{N} \in \mathbb{Z}^+ \), K \( \in \mathbb{R}^+ \) and a function \( \varphi : \mathbb{Z}^+ \rightarrow \mathbb{R} \), where

\[
\varphi(N) = \gamma + \frac{K}{N},
\]

and the social evaluation functions \( W_{CU} \) and \( W_{CLU} \) are restricted to a domain \( \overline{D} \subseteq \{(N, \upsilon) | N \in \mathbb{Z}^+, \upsilon \in \mathbb{R} \text{ for } N \leq \hat{N}, \upsilon \leq \varphi(N) \text{ for } N > \hat{N}\} \).

Proof: By definition a social evaluation function avoids RC-γ if and only if there exists a value \( \alpha > 0 \) and \( (N, \upsilon) \in D \) with \( \upsilon > \gamma \) such that, for all \( \varepsilon \in (0, \alpha) \) and all \( M > N \) with \( (M, \gamma + \varepsilon) \in D \),

\[
W(M, \gamma + \varepsilon) \leq W(N, \upsilon). \tag{11}
\]

Consider one such combination of \( \alpha, N \) and \( \upsilon \). Substituting from (9) into (11) for CLU yields

\[
M \varepsilon \leq N(\upsilon - \gamma). \tag{12}
\]

(CU is a special case of CLU with \( \gamma = 0 \).) Condition (12) is violated on the unrestricted domain \( D = (\mathbb{Z}^+ \times \mathbb{R}) \), since, for any positive value of \( \varepsilon \), the left-hand side can be made arbitrarily large by choosing an arbitrarily large value of \( M \). Therefore, (12) only holds on a restricted domain. In particular, for any given \( M > N \), (12) implies a maximum value of \( \varepsilon \). Define the function \( \tilde{\varepsilon} : \mathbb{Z}^+ \rightarrow \mathbb{R} \) such that \( \tilde{\varepsilon}(M) \) represents this maximum value. From (12)

\[
\tilde{\varepsilon}(M) = \frac{N(\upsilon - \gamma)}{M}. \tag{13}
\]
As indicated, condition (11) is to be tested for all values of representative utility given by $\gamma + \varepsilon$. Therefore, the addition of $\gamma$ to both sides of (13) will yield an upper bound on the values of representative utility for which (11) holds under CLU. Define the function $g : Z_{++} \to \mathbb{R}$ such that $g(M)$ represents this upper bound, i.e.,

$$g(M) = \gamma + \bar{\varepsilon}(M) = \gamma + \frac{N(v - \gamma)}{M}. \quad (14)$$

The requirement that (12) only holds for $\varepsilon \in (0, \alpha)$ limits the values of $M$ for which (14) must apply. Consider the value $\varepsilon = \alpha$, and define $\hat{M} \in Z_{++}$ such that $\hat{M} \alpha \leq N(v - \gamma)(\hat{M} + 1) > N(v - \gamma)$. Then for $N < M \leq \hat{M}$, the restriction $\varepsilon \in (0, \alpha)$ is sufficient to guarantee that (12) holds, without the upper bound restriction (14). In contrast, for $M > \hat{M}$, it is also necessary to impose (14).

Now observe that $g(M)$ has the same form as $\varphi(N)$ (with a change in population notation from $N$ to $M$), since $N(v - \gamma)$ is a constant in (14). Thus, necessity is proven.

To prove sufficiency, consider an arbitrary value $K \in \mathbb{R}_{++}$. There exist values $N \in Z_{++}$ and $v > \gamma$ such that $N(v - \gamma) = K$; in particular any combination satisfying $v = \gamma + K/N$. Consider any such combination $(N, v)$ and substitute into the function $\varphi$ (with a change in the general population label from $N$ to $M$):

$$\varphi(M) = \gamma + \frac{N(v - \gamma)}{M}, \quad M > \hat{M}. \quad (15)$$

Now by continuity there exists $\alpha > 0$ such that $(K/(\hat{M} + 1)) < \alpha \leq (K/\hat{M})$. Rearranging the second inequality and substituting in for $K$ yields

$$\hat{M} \alpha \leq N(v - \gamma). \quad (15)$$

On the restricted domain, $v \in \mathbb{R}$ for $M \leq \hat{M}$ and $v \leq \varphi(M)$ for $M > \hat{M}$. For $M \leq \hat{M}$, we must test (11) for $\gamma + \varepsilon \leq \gamma + \alpha$, which is equivalent to verifying (12) for all $\varepsilon \leq \alpha$. In this case, $M \varepsilon \leq \hat{M} \alpha$. It follows from (15) that $M \varepsilon \leq N(v - \gamma)$, which verifies (12). Now consider $M > \hat{M}$. In this case, we must test (11) for $\gamma + \varepsilon \leq \varphi(M)$, which simplifies to $M \varepsilon \leq N(v - \gamma)$. Once again (12) is verified. ■

The remainder of this section investigates the combinations of restrictions on utility and material consumption which can be employed to generate the upper bound $\varphi(N)$ required by Theorem 1. The search proceeds under three cases with regard to the values of neutral and subsistence consumption:

(i) $c_n \geq c_s > 0$, (ii) $c_n \geq c_s = 0$, (iii) $c_n > c_s = 0$ and all individuals have the same preferences.

Since representative utility is equal to average utility under CU and CLU, it follows that the existence of an upper bound on representative utility, for a
given population, entails an upper bound on total utility. Thus, it is necessary to identify conditions under which a maximum value of total utility exists. Total utility is defined

$$T \equiv \sum_{i=1}^{N} U_i(x_i).$$

The maximization problem is

$$\max \{ T \} \text{ subject to }$$

(i) \( N \in \mathbb{Z}_+ \) given, \( c_s \in \mathbb{R}_+ \) given,

(ii) \( U_i \in \xi \) for \( i \in \{1, \ldots, N\} \), and

(iii) \( \{x_1, \ldots, x_N\} \in X^N \).

Two preliminary results for quasi-linear utility will prove useful. Consider two individuals who must share a given quantity of the material consumption good. Both have utility functions in \( \bar{\xi}_\ell \) with the same lower-bound neutral consumption, \( c_n \). The first result establishes how total utility can be increased by reallocating material consumption between the two individuals when their utility functions have different slopes, \( b \).

**LEMMA 2:** For any two individuals \( i, j \in \mathbb{Z}_+ \), with utility functions \( U_i, U_j \in \bar{\xi}_\ell \) such that \( b_i > b_j \) and \( c_{n,i} = c_{n,j} = c_n \), and a fixed supply of material consumption to share, \( \bar{x}_1 \geq 2c_s \), the sum of utilities \( U_i + U_j \) is maximized by the corner allocation \( (x_{i1} = \bar{x}_1 - c_i, x_{j1} = c_s) \).

**Proof:** The sum of utilities is

$$U_i + U_j = b_i(x_{i1} - c_n) + b_j(x_{j1} - c_n)$$

$$= -(b_i + b_j)c_n + b_ix_{i1} + b_jx_{j1}. \quad (18)$$

Since \( b_i > b_j \), \( U_i + U_j \) is maximized by allocating as much of \( \bar{x}_1 \) as possible to individual \( i \). \( \square \)

The second result shows whether it is better in the setup above for the slopes to be the same or different. In the case with \( c_n > c_s \), the maximized sum of utility is greater when the utility functions have different slopes \( b \). In contrast, when \( c_n = c_s \), the maximized sum of utility is the same regardless of the slopes.
LEMMA 3: For any two pairs of individuals $i, j \in Z_{++}$ and $k, h \in Z_{++}$, with utility functions $U_i, U_j, U_k, U_h \in \xi_\ell$ such that $b_i = b_j = b_k > b_h$ and $c_{n,i} = c_{n,j} = c_{n,k} = \bar{c}_n$, and fixed supplies of material consumption, $\bar{x}_{ij} \geq 2\epsilon$, and $\bar{x}_{kh} \geq 2\epsilon$, to be shared between $(i, j)$ and $(k, h)$, respectively, with $\bar{x}_{ij} = \bar{x}_{kh}$. (i) the maximized sum of utility $u_k^* + u_h^*$ exceeds the maximized sum $u_i^* + u_j^*$ when $c_n > \epsilon$, and (ii) the maximized sum of utility $u_k^* + u_h^*$ equals the maximized sum $u_i^* + u_j^*$ when $c_n = \epsilon$.

Proof: Substituting $x_{i1} + x_{j1} = \bar{x}_{ij}$ into (18) yields

$$U_i + U_j = -(b_i + b_j)\epsilon_n + b_j\bar{x}_{ij}$$

since $b_i = b_j$. This value is independent of the allocation. For convenience, consider the corner allocation $(x_{i1} = \bar{x}_{ij} - \epsilon, x_{j1} = \epsilon)$. For the pair $(k, h)$, Lemma 2 indicates that the corner allocation $(x_{k1} = \bar{x}_{kh} - \epsilon, x_{h1} = \epsilon)$ maximizes the sum $U_k + U_h$. Thus we compare the two corner allocations. Since $b_i = b_k$ and $\bar{x}_{ij} = \bar{x}_{kh}$, it follows that $u_i^* = u_k^*$. Thus we need only compare $u_j^*$ and $u_h^*$. At the maxima, $u_j^* = -b_j(\epsilon_n - \epsilon) \leq -b_h(\epsilon_n - \epsilon) = u_h^*$. Since $b_j > b_h$, it follows that $u_j^* \leq u_h^*$ as $\epsilon_n \geq \epsilon$.

We now take up the three cases.

Case 1: $\bar{c}_n \geq \epsilon > 0$.

First, consider problem (17) when utility is restricted to $\xi_{\ell}^{b,\bar{b}}$, i.e., the class of quasi-linear functions with maximum and minimum slopes $\bar{b}$ and $b$, respectively. For this purpose, consider the utility profile $U^* = \{U_i = \bar{b}(x_{i1} - \epsilon_n), U_{j\neq i} = b(x_{j\neq i1} - \epsilon_n)\}$; i.e., one individual $i$ has maximal slope $\bar{b}$, everyone else has minimal slope $b$, and everyone’s utility is characterized by the common lower-bound value of neutral consumption, $\epsilon_n$.

LEMMA 4: If (i) the law of conservation of matter holds, (ii) resources are bounded, (iii) $U_i \in \xi_{\ell}^{b,\bar{b}}$ and monotonic for all $i \in \{1, \ldots, N\}$, (iv) neutral consumption is lower-bounded, and (v) $\epsilon_n \geq \epsilon$, then there exists a solution to problem (17). If $\epsilon_n > \epsilon$, the solution is unique and is given by the utility profile $U^*$ and the corner allocation $(x_{i1} = S(N, \bar{E}) - (N - 1)\epsilon, x_{j\neq i1} = \epsilon)$ in which as much of the material consumption good as possible is given to individual $i$. If $\epsilon_n = \epsilon$, there are multiple solutions, including $U^*$ and the corner allocation.

Proof: Bounded resources and the conservation law entail an upper bound on the supply of material consumption $S(N, \bar{E})$ given $N$. Monotonicity of preferences entails that a solution to (17), if it exists, will involve the consumption of the upper bound value $S(N, \bar{E})$.

Equation (6) indicates that quasi-linear utility decreases in the neutral consumption level, $\epsilon_{n,i}$. Therefore, a solution to (17), if it exists, will
 involve only functions in \( \hat{\xi}_{n}^{b,b} \), i.e., upper bound functions based on the common lower-bound value \( \xi_{n} \). From (7), \( \bar{U}_{i}(x_{1}) = b(x_{1} - \xi_{n}) \).

Consider any feasible allocation of \( S(N, \bar{E}) \) among individuals—i.e., \( \{x_{1}\}_{i=1}^{N} \) such that \( \sum_{i=1}^{N} x_{i} = S(N, \bar{E}) \)—and any arbitrary utility profile \( \{\bar{U}_{i}\}_{i=1}^{N} \) drawn from \( \hat{\xi}_{\ell}^{\bar{b},b} \). Given the profile, Lemma 2 indicates that as much as \( S(N, \bar{E}) \) as possible (i.e., all amounts in excess of \( c_{i} \)) should be re-allocated to the individual(s) with the highest slope.

Now consider changing the profile. Ceteris paribus the process of re-allocating \( S(N, \bar{E}) \) will lead to higher values of \( T \) for profiles in which some individuals have the maximal slope \( b \). For \( \xi = c_{s} \), Lemma 3 indicates that total utility is maximized when only one individual has maximal slope \( b \) and everyone else has minimal slope \( \bar{b} \), i.e., utility profile \( U^{*} \). Then Lemma 2 entails that \( T \) is maximized by the corner allocation \( (x_{1} = S(N, \bar{E}) - (N - 1)c_{s}, x_{j\neq i,1} = c_{s}) \). For \( \xi = c_{s} \), Lemma 3 indicates that it does not matter whether there is only one individual with maximal slope \( b \) or many. ■

Now consider problem (17) for utility in \( \hat{\xi}_{c}^{b,b} \), i.e., concave upper-bounded utility with bounds \( \bar{U}_{i} \) such that \( \bar{b} \leq (\partial \bar{U}_{i}(\xi_{n,i})/\partial x_{11})(x_{i1} - \xi_{n}) \leq \bar{b} \). The following result demonstrates that the solution to this problem is identical to the one just obtained for quasi-linear utility.

**LEMMA 5:** If utility functions are monotonic, an allocation \( \{x_{i}\}_{i=1}^{N} \) and profile \( \{U_{i}\}_{i=1}^{N} \) solves problem (17) under the restriction \( U_{i} \in \hat{\xi}_{c}^{b,b} \) for all \( i \in \{1, \ldots, N\} \) if and only if it solves the problem under the restriction \( U_{i} \in \hat{\xi}_{\ell}^{b,b} \) for all \( i \in \{1, \ldots, N\} \).

**Proof:** By monotonicity, if a solution to (17) exists on \( \hat{\xi}_{c}^{b,b} \), it will involve upper-bound functions \( \bar{U}_{i} \) and the lowest possible value of neutral consumption, \( \xi_{n} \); i.e., the solution will be found in \( \hat{\xi}_{c}^{b,b} \). Note further that by concavity, \( \bar{U}_{i}(x_{11}) \leq ((\partial \bar{U}_{i}(\xi_{n,i})/\partial x_{11})(x_{i1} - \xi_{n}) \), for all \( x_{i1} \in \mathbb{R}^{1} \); i.e., the function is everywhere equal to or below the tangent line defined by the derivative at \( \xi_{n} \). But this tangent line is nothing other than a quasi-linear function (7) with slope \( b \) equal to the derivative. Therefore, any solution to (17) in \( \hat{\xi}_{c}^{b,b} \) will occur in the subset \( \hat{\xi}_{\ell}^{b,b} \subset \hat{\xi}_{\bar{b}}^{b,b} \), i.e., it will involve an upper-bound quasi-linear function based on \( \xi_{n} \). Finally, any solution to (17) in \( \hat{\xi}_{\ell}^{b,b} \) is a solution in \( \hat{\xi}_{\ell}^{b,b} \) (i) by definition of a subset and (ii) since as observed in the proof to Lemma 4 there are no solutions in \( \hat{\xi}_{\ell}^{b,b} \) which are not in \( \hat{\xi}_{\ell}^{b,b} \). Thus necessity is proven. Sufficiency proceeds in the reverse manner. ■

Together Lemmas 4 and 5 identify the corner allocation \( (x_{11} = S(N, \bar{E}) - (N - 1)c_{s}, x_{j\neq i,1} = kc_{s}) \) and utility profile \( U^{*} \) as a solution to (17)
under the stated conditions. Substituting this solution into (16) yields the maximized value of total utility

$$T^* = \tilde{b}(S(N, \overline{E}) - (N - 1)c_s - \epsilon_n) + (N - 1)b(c_s - \epsilon_n).$$

(19)

Dividing both sides of the above equation by $N$ yields the maximized value of average utility, which under CU and CLU provides an upper bound on representative utility. Denote this upper-bound $\bar{u}(N) = (T^*/N).$ It can now be shown that the existence of $\bar{u}(N)$ is sufficient for CU and CLU to avoid the repugnant conclusion on $\xi_c,$ the set of concave utility functions.

**THEOREM 2:** If (i) the law of conservation of matter holds, (ii) resources are bounded, (iii) $U_i \in \xi_c$ and monotonic for all $i \in \{1, \ldots, N\}$, (iv) neutral consumption is lower-bounded, and (v) $\xi_n \geq c_s > 0,$ then CU and CLU avoid RC-$\gamma$.

**Proof:** First, prove the result on $\xi^{b,\tilde{b}}_c,$ i.e., concave upper-bounded utility with bounds $\overline{U}_i$ such that $\tilde{b} \leq ((\partial U_i(c_{ni}))/\partial x_i) \leq b.$ An upper bound function $\bar{u}(N)$ exists under these conditions, by Lemmas 4 and 5, and its value is given by $\bar{u}(N) = \frac{T^*}{N}$ with $T^*$ defined by (19). A sufficient (though not necessary) condition for $\bar{u}(N)$ to diminish in $N$ is that $T^*$ diminishes in $N.$ Rewriting (19) yields

$$T^* = \tilde{b}S(N, \overline{E}) - (\epsilon_n - c_s)(\tilde{b} - b) - [c_s(\tilde{b} - b) + \epsilon_n b]N.$$

The first difference is

$$T^*(N + 1) - T^*(N) = \tilde{b}[S(N + 1, \overline{E}) - S(N, \overline{E})] - [c_s(\tilde{b} - b) + \epsilon_n b].$$

By the conservation law $\lim_{N \to \infty} [T^*(N + 1) - T^*(N)] = -[c_s(\tilde{b} - b) + \epsilon_n b] < 0.$ Thus $T^*$ is diminishing and crosses the horizontal axis in $N.$

Given values of $\tilde{b}$ and $b,$ define the population value $\tilde{N}^b,\tilde{b} \in Z_{++}$ such that $T^*(\tilde{N}^b,\tilde{b}) \geq 0$ and $T^*(N) < 0$ for all $N > \tilde{N}^b,\tilde{b}.$ Since $T^*$ and $\bar{u}(N)$ have the same sign, it follows that $\tilde{u}(\tilde{N}^b,\tilde{b}) \geq 0$ and $\tilde{u}(N) < 0$ for all $N > \tilde{N}^b,\tilde{b}.$

Now consider any arbitrary value $K \in \mathfrak{S}_{++}$ and the corresponding function $\varphi(N)$ defined in (10). Then for all $N > \tilde{N}^b,\tilde{b}, \nu \leq \tilde{u}(N) < 0 < \varphi(N),$ which represents a subset of the restricted domain $\overline{D}$ required by Theorem 1. More immediately, the repugnant conclusion is avoided because $W_{CU}$ and $W_{CLU}$ are negative and decreasing in $N,$ for all $N > \tilde{N}^b,\tilde{b}.$

Now, prove the result on $\xi_c.$ If $\xi_n = c_s,$ equation (19) reduces to $T^*(N) = \tilde{b}[S(N, \overline{E}) - Nc_s].$ In this case, the definition of $\tilde{N}^b,\tilde{b}$ entails $(S(\tilde{N}^b,\tilde{b}, \overline{E})/\tilde{N}^b,\tilde{b}) \geq c_s$ and $(S(\tilde{N}^b,\tilde{b} + 1, \overline{E})/(\tilde{N}^b,\tilde{b} + 1)) < c_s.$ These two conditions indicate that the value of $\tilde{N}^b,\tilde{b}$ depends only on $S(N, \overline{E})$ and $c_s,$ i.e., there exists a value $\tilde{N} \in Z_{++}$ such that $\tilde{N}^b,\tilde{b} = \tilde{N}$ for all $\tilde{b}, \tilde{b} \in \mathfrak{S}_{++}.$ Therefore $\tilde{u}(N)$ exists and crosses the horizontal axis $(\nu = 0)$ at $\tilde{N}$ for all possible values of $\tilde{b}, \tilde{b} \in \mathfrak{S}_{++}.$
THEOREM 3: If (i) the law of conservation of matter holds, (ii) resources are bounded, (iii) \( U_i \in \xi_{\epsilon} \) and monotonic for all \( i \in \{1, \ldots, N\} \), (iv) neutral consumption is lower-bounded, and (v) \( \xi_n \geq c_s = 0 \), then CU and CLU exhibit RC-\( \gamma \).

Proof: Lemmas 4 and 5 apply in this case without modification; i.e., problem (17) admits a solution on \( \xi_{\epsilon}^{b,b} \) with the associated welfare value \( T^* \) given in (19) and \( \bar{v}(N) = \frac{T^*}{N} \). Substituting \( c_s = 0 \) into (19) yields

\[
T^* = b(S(N, E) - c_s) - (N - 1) b \xi_n.
\]

Now consider the problem on \( \xi_{\epsilon} \). Provided \( S(N, E) > c_s \), one can always adjust \( \bar{b} \) and \( b \) to make \( T^* \) (and therefore \( \bar{v}(N) \)) arbitrarily large. And indeed \( S(N, E) > c_s \), for any \( N \in \mathbb{Z}^+ \), since \( S(N, E) \) is non-decreasing.
The Repugnant Conclusion

in $N$. Therefore, there does not exist an upper-bound on representative utility on $\tilde{\xi}_c$. It follows from Theorem 1 that CU and CLU exhibit RC-$\gamma$.

Finally, Lemma 1 establishes that testing the result on $\xi_c$ with lower-bounded neutrality is equivalent to testing it on $\tilde{\xi}_c$. ■

**Case 3:** $\frac{c_n}{\bar{c}_s} > c_s = 0$ and all individuals have the same preferences
CU and CLU avoid the repugnant conclusion in this case.

**THEOREM 4:** If (i) the law of conservation of matter holds, (ii) resources are bounded, (iii) $U_i = U \in \xi_c$ and monotonic for all $i \in \{1, \ldots, N\}$, (iv) neutral consumption is lower-bounded, and (v) $\xi_n > c_s = 0$, then CU and CLU avoid RC-$\gamma$.

**Proof:** First consider the existence of a solution to problem (17) on the class of utility functions $\xi_{b,\bar{b}}$, that is, quasi-linear utility functions with slopes bounded by $\bar{b}$ and $b$.

Bounded resources and the conservation law entail an upper bound on the supply of material consumption $S(N, E)$ given $N$. Monotonicity of preferences entails that a solution to (17), if it exists, will involve the consumption of the upper bound value $S(N, E)$.

Because quasi-linear utility decreases in the neutral consumption level, $c_{n,i}$, a solution to (17) will involve only functions in $\xi_{b,\bar{b}}$, i.e., upper bound functions based on the common lower-bound value $c_{\bar{n}}$. The form of such functions is given in (7) as

$$U_i(x_{i1}) = b(x_{i1} - c_{\bar{n}}).$$

All individuals have the same utility function in this case and therefore differ only by the allocation of the material consumption good, $x_{i1}$. Substituting into (16) yields total utility

$$T = \sum_{i=1}^{N} b(x_{i1} - c_{\bar{n}})$$

$$= b \left( \sum_{i=1}^{N} x_{i1} - Nc_{\bar{n}} \right)$$

$$= b \left( S(N, E) - Nc_{\bar{n}} \right).$$

Provided $S(N, E) > Nc_{\bar{n}}$, $T$ is maximized for a given $N$ by choosing the function with maximal slope $\bar{b}$; i.e.,

$$T^* = \bar{b} \left( S(N, E) - Nc_{\bar{n}} \right).$$

This expression is equivalent to equation (19) for heterogeneous preferences in the case when $c_{\bar{n}} = c_s$. Thus, Theorem 2 applies mutatis mutandis. ■
5. Conclusion

The paper has argued for a revised definition of the repugnant conclusion based on the critical level rather than neutrality. The reasonableness of this definition follows from the analogous definitions of neutrality and the critical level, with one defined for the individual and the other for the planner. Under this definition, both CU and CLU exhibit the repugnant conclusion on an unrestricted domain. Therefore, in the comparison of CU and CLU (with positive critical level), it may be reasonable to regard CU as superior, since at least it satisfies Pareto-plus.

The paper then considered a restricted domain corresponding with six properties: bounded resources, the law of conservation of matter, concavity and monotonicity of preferences, positive subsistence consumption, and positive neutral consumption. CU and CLU were shown to avoid the repugnant conclusion on this domain.

These six properties are quite weak. Boundedness of resources and the law of conservation of matter represent universal physical laws, while concavity, monotonicity, positive subsistence, and positive neutral consumption represent highly plausible characterizations of preferences. Thus at most utilitarianism can be said to exhibit the repugnant conclusion on implausible domains. One could argue that such an outcome does not provide a compelling basis for rejecting utilitarianism.

Finally, it was shown that CU and CLU avoid the repugnant conclusion when the restriction on subsistence consumption is weakened to a value of zero, provided preferences are identical. The latter assumption is stronger and therefore less compelling. Nonetheless, it does correspond with the structure of many economic models.

References


