Real options in harvesting decisions on publicly owned forest lands (Revised)

M.C. Insley * and K. Rollins †‡§

July 30, 2003

Abstract

This paper develops a two factor real options model of the harvesting decision over infinite rotations with mean reverting stochastic prices using a rigorous Hamilton-Jacobi-Bellman methodology. The harvesting problem is formulated as a Linear Complementarity Problem which is solved numerically using a fully implicit finite difference method. This approach is contrasted with Markov Decision Process models commonly used in the literature. The model is used to examine a proposed investment in intensive forest management in Ontario's boreal forests. The value of a representative stand in the Romeo Malette forest is estimated assuming complete harvesting flexibility. This is contrasted with the value of the stand when regulations dictate a window of time during which harvesting must occur.

Keywords: optimal harvesting, real options, allowable cut restrictions, intensive forest management, Markov decision process, linear complementarity problem
Running Title: Real options in harvesting decisions

*Department of Economics,University of Waterloo, Waterloo, Ontario N2L 3G1, minsley@uwaterloo.ca, Phone: (519)888-4567, ext 2146, Fax: (519)725-0530

†Department of Applied Economics and Statistics, University of Nevada, Reno, Nevada, krollins@cabnr.unr.edu

‡The authors acknowledge with gratitude the financial support of the Living Legacy Trust in Ontario, Tembec Inc., the Agricultural Research Institute of Ontario, the Upper Great Lakes Environmental Research Network, and the Ministry of Natural Resources of Ontario. The opinions expressed herein represent the views of the authors, and not of the agencies that provided financial support.

§The authors would also like to thank numerous individuals who provided useful information and comments. These include Margaret Penner (Forest Analysis Ltd.), George Bruemmer, Al Stinson and Paul Krabbe (Tembec), Wayne Bell (Ontario Ministry of Natural Resources), Glenn Fox (University of Guelph), and Peter Forsyth (University of Waterloo). Any shortcomings or errors are solely the responsibility of the authors.

1 Introduction and Literature Review

A large share of the forest economics literature deals with the problem of optimal harvesting under uncertainty.[1] Certain features of the problem that have received particular attention are the implications of the volatile nature of wood prices relative to the costs of harvesting, and managerial flexibility in the context of irreversible harvesting decisions. In resolving issues posed by these features, forest economists have realized that accurate formulation and modeling of many timber-harvesting problems are unlikely to result in closed form solutions. Forest economics models have used a number of approaches to solving complex optimal harvesting problems, including Markov Decision Process (MDP) models and simulation. Recent approaches to modeling the tree harvesting decision under stochastic prices have increasingly drawn from the burgeoning finance literature on the valuation of financial and real options.[2] This literature emphasizes the importance of including any opportunities for managers to adjust harvesting plans in response to stochastic events as they unfold - opportunities that may be thought of as embedded options. Ignoring management options or flexibility will result in an incorrect valuation of a forestry investment. One of the purposes of this paper is to distinguish between these different approaches, and to describe the contributions of real options approaches.

The importance of valuing management options is not a new discovery of the real options literature. Forest economists have recognized this for at least three decades (Hool [20] and Lembersky and Johnson [31]). The contributions of the real options literature are in the development of powerful decision models and solution techniques that can offer improvements over those based on MDP models, a standard in the forest economics literature; and in formulation of the decision problem by explicitly recognizing the parallels between financial options, such as call options on a stock, and real options, which refer to the opportunities to acquire real assets. In forestry we can view the opportunity to harvest a stand of trees as a real option similar to an American call option that can be exercised at any time. The exercise price is the cost of harvesting the trees and transporting them to the point of sale. Embedded in the tree harvesting opportunity is the option to choose the optimal harvest time, based on wood volume and price, as well as the option to abandon the investment if wood prices are too low. The parallels between forestry and financial decisions allows forest economists to make use of models and methods developed in the finance literature to improve upon current approaches to model harvesting decisions under uncertainty.[3]

In this paper, we develop a model of the tree harvesting decision at the stand level under stochastic prices. We call our approach a real options analysis because the focus is on the importance of options embedded in the harvesting decision and we draw upon techniques from the finance literature. Prices are assumed to follow a mean reverting process (first order autoregressive) and the objective is to maximize the net present value of the investment

[1]For an overview of this literature, see recent bibliographies by Newman [39] and Brazee and Newman [6]

[2]Dixit and Pindyck [11] and Trigeorgis [54] provide a systematic reviews of the topic.

[3]Another innovation in the finance literature relevant to forest economics is contingent claims analysis, which allows one to avoid the conceptual difficulties of specifying an exogenous discount rate. However for a storable commodity it is still necessary to estimate a "convenience yield" or market price of risk (Trigeorgis [54], Dixit and Pindyck [11]). We do not use contingent claims analysis in this paper; the estimation of the market price of risk for a forestry investment will be the subject of future research.

over an infinite stream of future rotations. Drawing from the finance literature, the tree harvesting problem is specified as a Linear Complementarity Problem (LCP), which is solved numerically using a fully implicit finite difference approach.[4] This is a rigorous technique for which the existence and uniqueness of solutions follow from a large mathematics literature. It has guaranteed convergence with easily determined error bounds.[5] To our knowledge this technique has not been used previously to solve a multirotation optimal harvesting problem with autoregressive prices when land value is determined endogenously. Previous literature addressing this problem makes various simplifying assumptions such as assuming land value is exogenous. This particular simplification, for example, could cause difficulties if one is contemplating the impact of a policy change that would affect land value.

Almost thirty years ago, Lembersky and Johnson illustrated that a simplistic Faustmann type approach that maximizes expected present net value ignores managerial flexibility and the use of such a rule undervalues the resource. Speaking in the context of optimal actions for a forest manager faced with uncertainty in product prices, these authors point out that "(...) it is not advisable to predetermine the specific management action to carry out at each of the future time-points at which a decision will be made. A predetermined action can turn out to be inappropriate for stand and market conditions at implementation." The decision maker who has the ability to make decisions based on observed outcomes of random variables such as price and growth over time, instead of having to follow a prescribed rule based on expected values, can decide to harvest "early" to take advantage of an upswing in prices or delay harvesting if prices are depressed.[6] The value of an ability to delay an irreversible decision in order to observe and gain information on stochastic variables describes an option value (Arrow and Fisher [2]). While the concept of option value has been a part of resource economics problems for decades, the explicit formulation of the decision problem, and solution approaches have proven to be complex.

A number of studies have focused on general theoretical implications of the problem of uncertainty in harvesting decisions, using stylized analytical models with closed form solutions. These models necessarily must greatly abstract from the reality of most forest harvesting problems in order to obtain analytical solutions. So while this approach has been used to theoretically examine general attributes of the problem, it has not proven practical for most applications.

For example, Brock *et al* [8], Brock and Rothschild [7], Lohmander [33], and Miller and Voltaire consider the harvesting problem in the general context of stochastic capital theory and optimal stopping problems. This approach focuses on deriving analytical solutions and comparative static results to determine how the value of the asset varies with the parameters that describe the stochastic growth process of the state variables. The majority of these are single rotation models, while Miller and Voltaire [36] extend the Brock *et al* [8] capital theory model to the multi-rotation case and develop a barrier rule for stochastic revenue. These

[4]See Wilmott *et al* [58] and Tavella [50] for details of LCPs and numerical solution techniques.

[5]Wilmott [58] discusses American options in terms of variational inequalities and linear complementarity formulations. Proofs of existence and uniqueness of solutions are found in Elliott and Ockendon [13], Friedman [14], and Kinderlehrer and Stampacchia [30].

[6]This is true if prices follow a stochastic process that reverts to some mean over time. If prices follow geometric Brownian motion, then the value of flexibility comes from avoiding uneconomic harvests.

models treat revenue at the stochastic variable; the price and quantity state variables are not separately distinguished.

Willassen [56] uses the theory of stochastic impulse control to derive an explicit solution to the stochastic multirotational optimal harvesting problem with revenue alone as the state variable, not price and quantity separately. Drift and diffusion parameters of the stochastic process are independent of time. Sødal [46] also derives the closed form solution to the same problem using a simplified approach based on Dixit *et al* [12]. While allowing for closed form solutions, these models are not practicable for more realistic problems where price and quantity must be separately modeled.

Clarke and Reed [10] and Reed and Clarke [44] derive an optimal harvesting rule for the single rotation problem when price and growth are both random variables. Price is assumed to follow geometric Brownian motion whereas growth in volume is a function of stand age or size plus random Brownian motion. Under their (myopic look-ahead) rule the size at which a stand is harvested becomes a random variable that is independent of the absolute level of timber prices. This is a consequence of assuming geometric Brownian motion and ignoring costs. They also extend their rule to the multi-rotation problem. Insley [25] and Yin and Newman [59] discuss the impact of including harvesting and management costs in these models; in particular, that the optimal harvest time is no longer independent of price.

Using the results of Clark and Reed, but incorporating land rent costs as deterministic, Yin and Newman [61] compare the optimal harvesting time where growth and price are stochastic to what would be proscribed by a Faustmann, or maximum sustained yield, rule. In reality land rent should reflect the value of the bare land, which would equal the expected discounted net benefit from optimally managing the timber stand forever. Thus land rent should be endogenous, determined jointly with the value of the harvesting opportunity, although this considerably complicates the analysis.

Thomson [53] is an example of one of the earliest uses of real options methods in a forestry application. Thomson determines land rent endogenously assuming stumpage prices follow geometric Brownian motion. He compares stand value and rotation ages (as a function of price) with a fixed price Faustmann model. Thomson solves his model using a lattice method (a binomial tree), which is commonly used in finance and is in fact a simple version of an explicit finite difference scheme. (Wilmott [57] discusses the advantages of finite difference methods, such as the one used in this paper, over the binomial tree.)

Morck *et al* [37] more explicitly brings to their model insights from financial real options methods. They use a contingent claims approach to determine the optimal harvesting rate for a firm with a 10 year lease on a mature forest. This is a problem of inventory management where growth in inventory is assumed to follow Brownian motion with a drift, and timber prices are assumed to follow geometric Brownian motion.

In general, the assumption of geometric Brownian motion makes solution of the tree-harvesting problem more tractable. If management and harvesting costs are ignored, the problem can be solved analytically. It is recognized in the literature that an assumption of geometric Brownian motion is not realistic for commodities over the long term, because of the implication that the expected price level and variance will rise over time without bound. Mean reversion is thought to provide a better description of the price path of many commodities (see Schwartz [49], and references therein).

A number of studies examine the statistical properties of stumpage prices, including Haight and Holmes [16], Brazee *et al* [5], Hulkrantz [22], and Yin and Newman [60]. There is, in general, support that competitively determined stumpage prices in the markets studied exhibit autoregressive (mean reverting) behavior. Saphores *et al* [48] find evidence of both jumps and ARCH effects in stumpage prices in the U.S. Pacific Northwest.

Given mean-reverting prices, the optimal rotation problem must be solved numerically. Models based on MDPs represent one possible approach. MDPs have become a standard in the forest economics literature for incorporating ecological and market risks into harvest decisions. This approach models stochastic state variables in discrete time and computes matrices of transition probabilities that reflect the probability of moving from one state to another, conditional on management decisions. The transition matrix is typically estimated by simulation, and various techniques are used to determine optimal decisions based on the possibilities offered by the transition matrix. These techniques include the policy improvement algorithm, linear programming, and successive approximation, and are described in the operations research literature (see, for example, Hillier and Lieberman [19]).

The MDP approach is somewhat limited by the use of the transition matrix, which expands dramatically as the number of possible outcomes of a stochastic variable increases. This is typically handled by grouping stochastic outcomes, such as prices, into a small number of categories, such as ‘high’, ‘medium’ and ‘low’. As will be shown below, models based on real options approaches do not require simulations to calculate transitions probabilities, and stochastic variables can be specified in much finer detail. This allows for finer degrees of resolution, and therefore significantly greater accuracy in the determination of decision criteria. However, it may be noted that if there are many stochastic factors, MDP models (or simulation, discussed below) may be the only computationally feasible option.

Examples of MDP models are numerous and include Lembersky and Johnson, Norstrom [40], Kao [26], Teeter and Caulfield [52], and Kaya and Buongiorno [29], Lin and Buongiorno [32] and Buongiorno [9]. Lin and Buongiorno incorporate natural catastrophes, as well as diversity of tree species and tree size. Buongiorno [9] provides useful insight by interpreting Faustmann’s formula as a special case of a MDP model in which the probability of moving from one state to another is equal to unity.

Plantinga [43][7], Haight and Holmes [16], and Gong [15] solve the optimal harvesting problem with mean reverting prices using a Markov transition matrix and a discrete stochastic dynamic programming algorithm. For simplicity, these papers treat the value of the bare land as deterministic. Brazee *et al* [5], in examining the benefit of adaptive management when prices are mean reverting, note that the gains vary directly with the level of mean reversion.

Gong [15] compares a stochastic dynamic programming approach, based on use of a MDP, with a simulation method for finding the optimal harvest policy. He notes the difficulty of determining accurate stand values with simulation. Longstaff and Schwartz [34] and Anderson [1] have recently proposed two methods to adapt a simulation approach to valuing an American option.[8] This is the case for the option to harvest a stand of trees as there is a

[7] Plantinga provides a helpful conceptual linking of the notion of option value with the previous literature on harvesting under uncertainty

[8] Hull [21] provides a useful summary of these methods

range of ages over which harvesting may be optimal depending on lumber prices of the day. Wilmott [57] discusses in more detail the difficulties of using simulation approaches to value an American type option - i.e. one for which early exercise may be optimal.

As a contribution from the real options literature, an alternative to MDP approaches and simulation is to formulate the problem in terms of a partial differential equation, which can be solved numerically using techniques from the large literature on numerical analysis. Saphores *et al* [48] demonstrates the use of Galerkin's method (a finite element method) to solve the problem of whether to preserve or harvest a stand of old growth forest when lumber prices follow geometric Brownian motion with jumps. Insley [25] formulates the tree harvesting problem with mean reverting prices for a single rotation as a Linear Complementarity Problem (LCP), and demonstrates a numerical solution using a fully implicit finite difference scheme.[9]

This paper extends the model of Insley [25] to a multi-rotation framework under mean reverting prices with the bare land value determined endogenously. A linear complementarity problem (LCP) formulation completely specifies the optimal decision process. Much of the previous cited literature may be thought of in terms of different approaches to solving this LCP. Unlike the single rotation problem, the multi-rotation case represents a "path dependent option", as the optimal harvesting age today depends on decisions about harvesting made in previous years. Path dependency complicates the solution (see Wilmott [57]). In this paper we demonstrate a fairly straightforward solution technique, which to our knowledge has not appeared before in the literature.

The solution of the LCP can be directly linked to the transition probability matrix estimated in MDP models. We will show that the MDP approach is, in fact, an indirect method of solving the LCP and that it is unnecessary to use simulation to calculate transition probabilities. The transition probability densities are the solutions of the forward Kolmogorov equations, which are embedded in the solution of the LCP. The explicit connection between the MDP model and the numerical solution of the LCP is discussed in more detail in Section 9.

We use the model to evaluate policy issues that have arisen recently in Ontario's forestry sector, including the economics of more intensive forest management and the impact of harvesting restrictions on the economics of forestry. The detailed results presented in this paper illustrate the gains in accuracy from using the full LCP rather than a MDP approach.

In summary the contributions of this paper are threefold.

- We use a rigorous Hamilton-Jacobi-Bellman approach to formulate a multi-rotation optimal harvesting problem with mean reverting prices as a Linear Complementarity Problem (LCP) in the spirit of the financial real options literature. We demonstrate its numerical solution using a finite difference method that allows us to specify the required level of accuracy. To our knowledge this represents a new contribution in the forestry economics literature.

- We demonstrate the precise relationship between the LCP approach and the more traditional MDP models. In this way we show how the use of methods developed in

[9] Wilmott *et al* [58] discusses LCPs as well as numerical methods for solving option value problems.

the finance literature can improve upon the solution to the forest harvesting decision problem.

- We use the model to consider issues in the Ontario forest sector including harvesting restrictions and the push to undertake more intensive forest management.

2 Policy Application

The model developed here is used to examine several practical policy questions that have arisen in Ontario's provincially owned forests, concerning proposals to engage in *intensive forest management* (IFM) in designated areas. While there is no generally accepted definition of the term 'intensive forest management', it typically implies greater spending on silvicultural practices in order to increase wood volume and/ or value. Density regulation and genetic improvement are involved. Sometimes IFM is also defined to include management for multiple objectives rather than just timber. In this paper we define three different levels of forest management based solely on the level of silvicultural expenditures as given in the policy context.

IFM has been suggested in Ontario government and industry circles as a means of maintaining flows of wood to mills while reducing the land base eligible for harvesting. IFM requires a significant financial investment when a stand is young, in order to obtain increased yields when the stand matures. Given uncertain wood prices and an up-front irreversible investment, the economics of this type of investment should be looked at using an approach that accounts for the flexibility to manage for price risk. The value to a firm of an investment in IFM will be maximized if the firm has complete flexibility to delay or accelerate harvests in response to changing lumber prices.

Firms with licenses to harvest in Ontario's public forests are constrained by allowable cut regulations, and hence are limited in their ability to manage for price risk. Such constraints reduce the return to the firm from carrying out IFM. The real options model developed here is used to compare the value of an investment in IFM with and without restrictions on harvesting time. The difference between the two gives some indication of the cost of current restrictions.

Almost 90 percent of the timber volume consumed by Ontario mills in 2000-2001 was from Ontario Crown land.[10] There is also an active private timber market, mainly in the southern portion of the province, which provided about 10% of timber volumes in that same year.[11]

Ontario's current regulatory regime dates from the Crown Forest Sustainability Act of 1994 which transferred much of the responsibility for forest management activities to industry. The major players in the industry are vertically integrated firms with their own lumber and pulp mills. They operate under 20 year renewable Sustainable Forest Licenses, which

[10] Huennemeyer [23] and Ontario [42]provide an overview of Ontario government forest management policies.

[11] The Ontario Ministry of Natural Resources commissioned a survey of the provinces private woodlots, which was released in 2001, RISI [45]

grant the right to harvest within a certain defined area under an approved Forest Management Plan and are subject to audit every five years. The licensees are responsible for the costs of preparing and implementing management plans, including managing for non-timber forest values on their allotted lands, and covering the costs of the 5-year mandatory audits. Provincial responsibilities include approving forest management plans and carrying out spot checks of firms' self-reporting. Failure to manage according to the approved management plans may result in financial loss for the firm, through loss of licenses or other penalties.[12]

The amount that a license holder can harvest each year is based on an Available Harvest Area, which according to the Ministry of Natural Resources, is set at levels that maintain a healthy forest while attempting to achieve a constant volume of wood flowing at regional levels. License holders are expected to stay reasonably close to their allowable cuts. At one point in Ontario's history, a firm risked forfeiting its harvesting license if it failed to harvest the allowable cut. In recent years it appears that firms are not penalized for harvesting less than the allowable cut, and there is little documented evidence of firms exceeding the allowable cut. However, firms are affected by current industry structure, developed over years of forestry regulation, which tends to favor even wood flows.

The desire to maintain even wood flows runs contrary to the notion of managing harvests in response to price volatility. Ideally the benefits of varying harvests in response to price would be balanced against the costs to the mill (and to the communities which may depend on a mill for employment) of an uneven wood flow. This paper addresses only one side of this balance - the value of the stand when harvesting is responsive to price.

In 1999 the Ontario government undertook an initiative to expand the area of provincial forest reserved as parks and protected areas by 2.4 million hectares. At the same time it was promised that the expansion of parks would not reduce the long-term supply of wood fibre to mills or increase the cost of that wood.[13] One suggestion to maintain fibre flow to mills was the establishment of Enhanced Wood Supply Agreement areas on which IFM could be undertaken. The premise was that IFM would enhance the quantity and quality of wood in grown designated areas so that fibre flow to mills could be maintained even while increased acreage is set aside as parkland (Accord [4]). Industry and provincial representatives have noted that the economics of IFM investments may be inadequate to induce private firms to undertake the necessary investment and that some sorts of incentives may be needed.

To fully address the efficiency and other economic implications of recent Ontario government policy initiatives would require consideration of all benefits and costs of these policies, including environmental impacts of increases in protected areas and possible effects on employment in logging communities. This would require modeling the optimal harvesting decisions at the forest level, rather than the stand level as is done in this paper. Such a complete analysis is beyond the scope of this paper. Our goal is more limited, but is an important building block to addressing the larger questions. Using a stand level model and a real options approach, we estimate the value of the commercial harvest and the opportunity

[12]Huennemeyer and Rollins [24] discuss the incentive effects generated by private management of public and private forest benefits. It is beyond the scope of this paper to assess the implications of this system on social and private returns to forestry. However, it is clear that the returns to commercial forestry operations reflect social as well as private objectives.

[13]Documents that describe these recent policy initiatives include [4] and [41].

cost of policies to maintain sustained yields, such as the promotion of IFM.

3 Formulation of the Model

The tree harvesting decision is initially modelled from the point of view of a social planner. Hence taxes and stumpage payments are ignored. The price of timber sold to the mill is assumed to follow a known stochastic process. The value of the stand of trees is estimated assuming the harvesting decision will be determined optimally in the future whatever the price path turns out to be. In a world without taxes and stumpage payments, the estimated value at the beginning of the first rotation is the maximum amount that a private firm would be willing to pay for the right to harvest the trees at some time in the future, providing the firm has complete flexibility to determine the harvest date and that markets exist for the logs.

It will be assumed that the price of saw logs, P, is a mean reverting stochastic process, an assumption that makes intuitive sense for many commodities (Schwartz [49]). When a commodity's price is relatively high we would expect production to increase as more suppliers enter the market putting downward pressure on price. The mean reverting level would be expected to reflect long run average production costs.

The mean reverting price process is specified as follows:

$$dP = \eta(\bar{P} - P)dt + \sigma P dz \tag{3.1}$$

where

$$\begin{aligned} P &= \text{the price of saw logs} \\ \eta &= \text{mean reversion parameter} \\ \sigma &= \text{the constant variance rate} \\ dz &= \text{increment of a Wiener process} \end{aligned}$$

According to Equation (3.1), price reverts to a long run average of $\bar{P}$, but the variance rate grows with P. The variance rate grows with P, so that the variance is zero if P is zero. This format is more appealing than the simple Ornstein-Uhlenbeck process in which the variance rate is σdz. In the simple Ornstein-Uhlenbeck process, as price becomes small, the constant volatility could cause prices to become negative.

Wood volume, Q, is assumed to be deterministic and depends on the time since the last harvest:

$$Q = g(\alpha); \quad \alpha \equiv t - t_h \tag{3.2}$$

where t is the current time and t_h is the time of the last harvest. It follows that:

$$dQ = g_\alpha dt \tag{3.3}$$

Assuming that Q is a non-decreasing function of α[14] then the inverse function $\alpha = \alpha(Q)$ exists, so that

$$g_\alpha = g_\alpha(\alpha) \equiv \phi(Q). \tag{3.4}$$

Equation (3.3) can then be written

$$dQ = \phi(Q)dt \tag{3.5}$$

The decision to harvest the stand of trees can be formulated as an optimal stopping problem where the owner must decide in each period whether it is better to harvest immediately or delay until the next period. This decision process can be expressed as a Hamilton-Jacobi-Bellman equation:

$$V(t, P, Q) = max\left\{(P - C)Q + V(t, P, 0); A + (1 + \rho)^{-1} E[V(t + \Delta t, P, Q)]\right\} \tag{3.6}$$

where

E = expectaton operator
V = value of the opportunity to harvest
P = price of saw logs
C = per unit harvesting cost
Q = current volume of timber
A = per period amenity value of standing forest less any management costs
ρ = annual discount rate

The first expression in the curly brackets represents the return if harvesting occurs in the current period, t. It includes the net revenue from harvesting the trees plus the value of the land after harvesting, $V(t, P, 0)$. This is the value that could be attained if the land were sold subsequent to the harvest, assuming that the land will remain in forestry.

The second expression in the curly brackets is the *continuation region* and represents the value of delaying the decision to harvest for another period. It includes any amenity value of the standing forest, such as its value as a recreation area, less any forest management costs, A. It also includes the expected value of the option to harvest in the next period, discounted to the current period.

Following standard arguments (Dixit and Pindyck [11], Wilmott *et al* [58]) we can derive the following partial differential equation that describes V in the continuation region.

$$V_t + \frac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P)V_P - \rho V + A + \phi V_Q = 0 \tag{3.7}$$

Equation (3.7) describes the behaviour of V in the continuation region of Equation (3.6) when it is optimal to continue holding the option to harvest. The full optimal stopping

[14] For some regimes after age 95 volumes are expected to decline somewhat due wood deterioration at advanced ages. Our model is written so that we require that a unique age be associated with each volume level. To meet this requirement volumes were kept constant after age 95, rather than being allowed to decline.

problem can be formulated as a LCP, which is equivalent to the optimal stopping problem of Equation (3.6) (Wilmott [58], Tavella [50]). T denotes the terminal time. Let τ be defined as time remaining in the option's life, i.e. $\tau \equiv T - t$. Rearranging Equation (3.7) and substituting τ for t, we define an expression, HV, as follows:

$$HV \equiv \rho V - \left[\frac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P)V_P + A + \phi V_Q - V_\tau\right] \tag{3.8}$$

Then the LCP is:

$$\begin{aligned} &(i) \quad HV \geq 0 \\ &(ii) \quad V(\tau, P, Q) - [(P - C)Q + V(\tau, P, 0)] \geq 0 \\ &(iii) \quad HV\Big[V(\tau, P, Q) - [(P - C)Q + V(\tau, P, 0)]\Big] = 0 \end{aligned} \tag{3.9}$$

The LCP expresses the rational individual's strategy with regards to holding versus killing the option to harvest the stand of trees. ρV represents the return required on the investment opportunity for the rational investor to continue to hold the option. The expression within box brackets in Equation (3.8) represents the actual return over the infinitesimal time interval dt. The actual return has terms reflecting how V changes with changes in P and Q. It also includes the flow of amenity value less management costs, A. Part (i) of Equation (3.9) states that the required return for holding the option must be at least as great as the actual return. We would not expect a situation in which the required return is less than the actual return to persist in competitive markets. Part (ii) states that the value of the option, V, must be at least as great as the return from harvesting immediately. The return from harvesting immediately is the sum of the net revenue form selling the logs $(P - C)Q$ plus the value of the land after immediately after harvesting, $V(t, P, 0)$. V would never drop below the return from harvesting immediately, because the rational investor would harvest before that could happen. Finally, Part (iii) states that at least one of statements (i) or (ii) must hold as a strict equality. If $HV = 0$, it is worthwhile to continue to hold the asset. If $V - (P - C)Q - V(t, P, 0) = 0$ it is worthwhile harvesting the asset. If both expressions hold as strict equalities then the investor is indifferent between harvesting and continuing to hold the asset.

The LCP is solved numerically as is described in the Section 8. This involves discretizing the relevant partial differential equation including a penalty term that enforces the American constraint (Equation (3.8), ii). We are left with a series of nonlinear algebraic equations which must be solved iteratively. The implied Markov matrix can be found through manipulation of the discretized version of Equation (3.10). Details are provided in Section 9.

Boundary conditions for the problem are specified below.

1. **As $\boldsymbol{P \rightarrow 0}$**, we need no special boundary condition to prevent negative prices. Referring back to Equation (3.1), we see that as $P \rightarrow 0$, $dP \rightarrow \eta\bar{P}$ which is positive.

2. **As $\boldsymbol{P \rightarrow \infty}$**, we follow Wilmott [57] and set $V_{PP} = 0$.

3. **As $Q \to 0$**, we require no boundary condition since $\phi(Q) \geq 0$, as $Q \geq 0$, and the PDE is first order hyperbolic in the Q direction, with outgoing characteristic in the negative Q direction.

4. **As $Q \to \infty$**, we assume $\phi(Q) \to 0$, and hence no boundary condition is required.

5. **Terminal condition.** As T gets large it is assumed that $V = 0$. T is made large enough that this assumption has a negligible effect on V today.

4 Data and Parameter Estimates

The case examined in this paper is for a stand of Jack Pine (Site Class 1) in the Romeo Malette Forest Unit which is managed under a Sustainable Forest License by Tembec Inc. The Romeo Malette forest consists of 477,109 hectares of productive forest land and is located northeast of the town of Timmins, Ontario. Tembec is in the process of examining the prospects for intensive forest management in the unit, and as part of this process has invested in the development of improved growth and yield curves. The levels of forest management have been classified as being extensive, basic, or intensive, based on the degree of silvicultural investment:

- Extensive: Natural regeneration

- Basic: Assisted natural and artificial regeneration, including site preparation and removal of competing species. The focus is on manipulating species composition and achieving full site occupancy.

- Intensive: Assisted natural regeneration and artificial regeneration, featuring density regulation at a young age. As with basic management, the focus is to manipulate the species composition and achieve full site occupancy. In addition tree density is controlled to optimize individual tree growing space.

The silvicultural costs involved with each level of management are given in Table 1 (as estimated by Tembec). Tembec is also examining additional levels of intensive forest management, but for this paper only extensive, basic, and intensive are considered. Amenity value is assumed to be zero, so that A in Equation (3.8) reflects only silvicultural costs.

Yield curves for Jack Pine consistent with these management regimes in the Romeo Malette Forest Unit were provided by Tembec.[15] Net merchantable volume was divided into different product volumes based on prices that will be paid at the mill. Four different product groups have been identified as shown in Table 2. SPF1 and SPF2 receive the best prices at the mill. SPF3 is the residual that will be used for pulp. POBW are species that are harvested by regulatory requirement, even though their harvesting costs exceed their value at the mill.

Figure 1(a) shows the volumes of SPF1 and SPF2 for the different levels of management. Under intensive and basic management levels the higher valued products appear earlier and

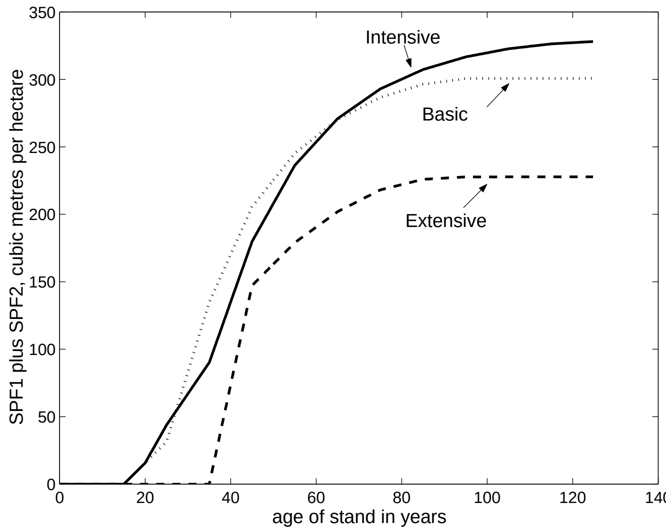
[15] The yield curves were estimated by M. Penner of Forest Analysis Ltd., Huntsville, Ontario, for Tembec.

Cost, $/ha	Extensive	Basic	Intensive	Timing (age)
Site preparation	0	200	200	1
Nursery stock	0	360	0	1
Improved stock	0	0	390	1
Planting	0	360	360	2
First tending	0	120	120	5
Second tending	0	0	120	8
Precommercial thinning	0	0	500	20
Data Management/Monitoring	5	10	20	35
Commercial thinning	0	0	200	35

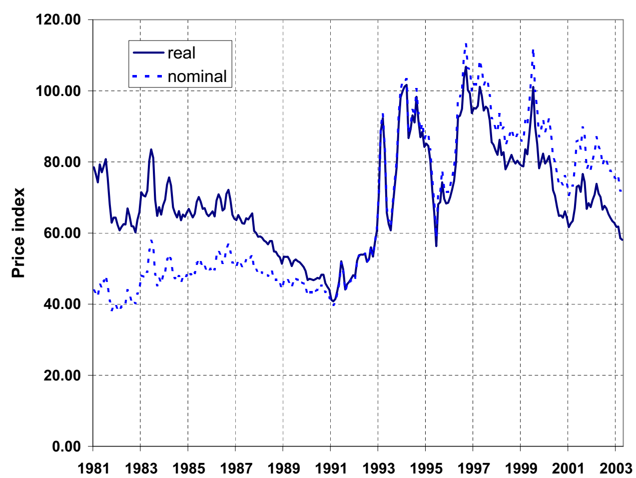
TABLE 1: *Silvicultural Costs*

SPF1 (spruce, pine, fir)	logs greater than 16 cm at small end
SPF2	12 - 16 cm at small end
SPF3	10 - 12 cm at small end
POBW	poplar, birch and other species

TABLE 2: *Merchantable Products*



(A) *Yield per hectare of high valued products under different management intensities*



(B) *Ontario softwood lumber price indices, Statistics Canada Cansim II Series V1575012, 1997=100 for nominal price index, Real price index is the nominal index deflated by the CPI.*

FIGURE 1: *Yields and prices*

Variable	Coefficient	t-statistic	Details:
$c(1)$	3.206	2.813	Sample: 1981:03 to 2003:04
$c(2)$	-.0470	-2.917	Number of observations: 266
$c(3)$	0.283	4.785	$H_0 : \delta = 0$
			$R^2 = 0.097$; SE of regression: 3.75
			Dickey-Fuller Critical Values:
			1% -3.4566; 5% -2.8725; 10% -2.5726

TABLE 3: *Augmented Dickey-Fuller test on the Ontario softwood lumber price index (Statistics Canada, Cansim II Series V1575012) deflated by the monthly consumer price index.*

reach greater volume than under extensive management. Details of the yield curves for all products are available from the authors upon request.

We did not have a historical time series for mill gate lumber prices, and hence used a price index for Ontario softwood lumber to estimate the parameters of the price process.[16] The index is shown in Figure 1(b). A correlogram of the data showed a very high autocorrelation coefficient of 0.96 at one month lag, tailing off gradually to reach 0.457 at a 15 month lag. This is an indication that the process is stationary.

A further test of stationarity is the unit root test (see Hamilton [17]), We performed an augmented Dickey-Fuller test by running the following regression:

$$P_t - P_{t-1} = c(1) + c(2)P_{t-1} + c(3)(P_{t-1} - P_{t-2}) \tag{4.1}$$

The results are shown in Table 3. The null hypothesis is that the data contain a unit root ($H_0 : c(2) = 0$) so that the true data generating process is a random walk (and hence non-stationary). We can reject the null hypothesis at the 5% significance level but not at the 1% level.

The augmented Dickey-Fuller test was used because of evidence of serially correlated error terms in the (non-augmented) Dickey-Fuller test. However evidence of serial correlation can also indicate the existence of autocorrelation in the variance of the disturbances or the ARCH effect. An ARCH text was done on the residuals of the regression:$P_t - P_{t-1} = c(1) + c(2)P_{t-1}$ There was strong evidence of the ARCH effect. This implies that an ARCH/GARCH model may be appropriate, however this would require a three-factor option pricing model with deterministic quantity, stochastic price and stochastic volatility. Such a model would be much more difficult to solve and is beyond the scope of this paper.

For this paper we assume lumber prices follow a mean reverting process as is described in Equation (3.1). A discrete time approximation is:

$$P_t - P_{t-1} = \eta \bar{P} \Delta t - \eta \Delta t P_{t-1} + \sigma P_{t-1} \sqrt{\Delta t} \epsilon_t \tag{4.2}$$

where ϵ_t is $N(0, 1)$. Dividing through by P_{t-1} and using the notation

$$c(1) \equiv \eta \Delta t;\; c(2) \equiv \frac{\eta \Delta t \bar{P}}{P_{t-1}};\; e_t \equiv \sigma \sqrt{\Delta t} \epsilon_t, \tag{4.3}$$

[16]The price index used was Statistics Canada monthly Industrial Product Price Index, V1575012, deflated by the consumer price index.

Variable	Coefficient	t-statistic	Details:	
$c(1)$	-.0272	-1.75	Sample:	1981:02 to 2003:04
$c(2)$	1.82	1.81	Number of observations:	267
			$R^2 = 0.012$	SE of regression: 0.052

TABLE 4: *Parameter estimates of Equation 4.4*

The relevant parameters can be estimated by ordinary least squares on the following equation:

$$\frac{P_t - P_{t-1}}{P_{t-1}} = c(1) + c(2)\frac{1}{P_{t-1}} + e_t \tag{4.4}$$

Regression estimates are shown in Table 4. From the definitions of $c(1), c(2)$ and e_t in Equation (4.3) and given that Δt is one month, the parameter estimates are $\eta = 0.33$, $\bar{P} = 67$, and $\sigma = 0.18$.

Clearly the mean reverting process we have chosen is not the best description of the historical data. It has been argued that with the rise of plantation-type forestry worldwide and the depletion of old growth forests, we may expect lumber prices to tend toward a path of mean reversion in the future.[17] It is the authors' opinion that as an economic model of a commodity price, the mean reverting model is an improvement over previous models which have assumed geometric Brownian motion.[18] An area of future research will be to solve the tree harvesting problem in the presence of ARCH effects and /or the presence of jumps.[19].

Harvesting costs and product prices were provided to the authors on a confidential basis. Representative harvesting costs for Ontario are reported in Rollins *et al* [47] as $31/$m^3$. The price of saw logs at the millgate is approximately $50/$m^3$.

This $\bar{P}$ estimate given above refers to the real price index; this had to be translated into a price to the mill in $/cubic metre. Because the recent price index values are close to $\bar{P}$, the current price going in to the mill reported (in confidence) by Tembec was chosen as $\bar{P}$ for the analysis. We use 3% and 5% real discount rates for the analysis.

5 The Economics of the Harvesting Opportunity

5.1 Flexible Harvesting Time

This section presents the estimated value of the option to harvest the representative stand of trees on public forest land (V in Equation (3.9)) given that the goal is to maximize the net present value of the commercial value of the timber. Consumer surplus is ignored, implying the wood harvested is destined for the export market. We assume that adequate log markets exist, and ignore any values other than commercial timber value. The decision variable is whether to harvest the stand given the market price for any given time period. This implies that there are no restrictions (regulatory or otherwise) on firms as to when a stand could

[17] Insley [25] discusses this issue

[18] Schwartz [49] provides a description of some alternate price models for commodities.

[19] See Saphores *et al* [48] for a discussion of testing for ARCH and jump effects as well as the difficulties of solving for the optimal decision in the presence of ARCH effects

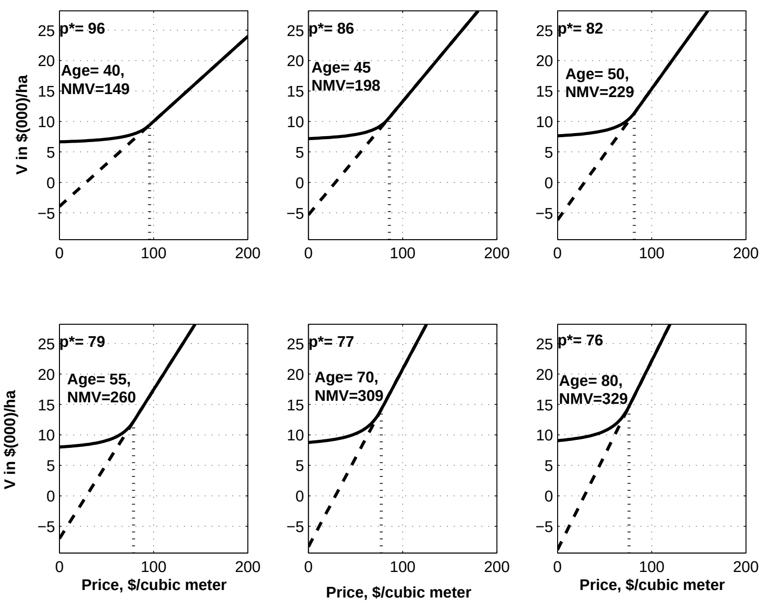


FIGURE 2: *Value of the opportunity to harvest a stand at different stand ages, discount rate = 3%, intensive treatment*

be harvested. We call this the social perspective to distinguish it from the perspective of a regulated firm which is constrained as to harvesting times. This implies that the values estimated in this model may not be achievable for a stand that is part of a forest unit managed so as to maintain an even annual flow of wood, since such a management goal would imply restrictions limiting the choice of when to harvest.

One purpose of the study is to evaluate the economics of proposed IFM options. Some of these options result in negative net present values. It was therefore necessary to constrain the model to disallow harvesting for the first 35 years until all silviculture expenditures had been made. Without this restriction the model would, in some instances, choose to harvest at a very young age to avoid large silviculture expenses in the future.[20]

Solving Equation (3.9) using the prices, costs and discount rate given in the previous section, we estimate, for any given stand age, the threshold price above which it is optimal to harvest the stand. Section 8 (Appendix 1) provides details about the solution of the numerical model. Figure 2 illustrates these results for a real discount rate of 3% under the intensive treatment regime, which involves significant management expenditures. Each individual graph in Figure 2 represents a different stand age, and indicates the net merchantable volume per hectare (NMV) achieved by that age, as well as the critical price, P^*, at which it is worthwhile harvesting. The solid curve in each of the graphs represents the value of the opportunity to harvest the stand of trees, V, if there are no restrictions on harvesting after the silvicultural treatments are completed, and markets are available for the logs. The dot-

[20]One rationale that has been proffered for IFM is that it makes possible higher levels of non-timber benefits in other forested areas, thereby increasing the overall value of the public owned forest to society. While this analysis does not include non-timber values, estimation of the magnitudes of both gains and losses from IFM would be useful for a full evaluation of these practices.

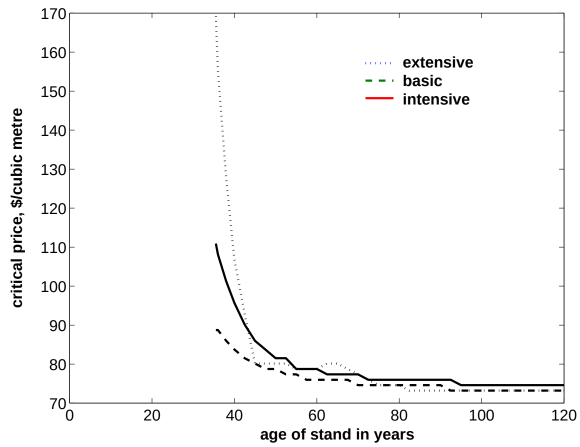


FIGURE 3: *Critical Price Versus Stand Age for Different Management Intensities, discount rate = 3%,*

ted line represents the payout from harvesting immediately. The dashed vertical line marks the critical price. When the value of the opportunity to harvest, V, is above the payout line, the value of delaying the harvest exceeds the value of harvesting immediately, and it is worthwhile waiting. Once V touches the payout line, it is worthwhile harvesting. The point of tangency determines the critical price (and demonstrates the smooth pasting condition.)

For a 40 year old stand, according to Figure 2, it would be worth harvesting if the price of SPF1 reached $\$96/m^3$. Although this is a fairly young age by boreal forest standards, with a mean reverting price, it makes sense to harvest early to take advantage of a temporarily high price. By age 45 the critical price has dropped significantly to \$86. As the stand ages the critical price continues to drop because the opportunity cost of delaying the harvest is reduced. The critical price reaches a steady state as growth slows.

Figure 3 depicts critical price versus stand age for our representative stand compared across the different management intensities. In the early years we observe that the critical prices for the extensive treatment are much higher than for basic or intensive. This is because growth is delayed in the extensive case, meaning that the opportunity cost of harvesting in these early years is much higher. Recall that the extensive treatment involves natural regeneration with no spending on planting or tending.

Of significant interest is the value of the opportunity to harvest the stand of trees at the beginning of the first rotation. This value represents the maximum amount that a firm would be willing to pay for harvesting rights to the stand, ignoring taxes and other charges, and assuming the firm has complete flexibility as to the timing of the harvest and that markets exist for the logs. A comparison of value over the different management intensities and for two different discount rates is shown in Table 5. Also shown in the table are values computed using the Faustmann formula, assuming a constant price equal to the long-run mean reverting level used in the stochastic model. The magnitude by which the real options approach exceeds the Faustmann analysis reflects the value of having the ability to optimally manage in the face of price volatility.

Both the Faustmann and the real options approaches indicate that the extra expenditures

3% discount rate	Real options	Faustmann
extensive	$1660	$890
basic	$1652	$305
intensive	$787	-$448
5% discount rate	Real options	Faustmann
extensive	$505	$299
basic	-$57	-$512
intensive	-$551	-$943

TABLE 5: *Value of the land at the beginning of the first rotation, $/hectare*

incurred in the basic and intensive regimes cannot be justified by the additional revenue that will accrue from harvesting the trees. There may be other reasons to incur these expenditures, (like preventing mill closures) but clearly the added silvicultural costs far outweigh the extra revenues received from achieving SPF1 volumes sooner and in greater quantity.

The analysis is very sensitive to the discount rate chosen, since the costs of IFM occur in the first 35 years, and the benefits, in terms of higher volumes, occur after 35 years. The volumes for intensive management are actually lower than for basic management until about age 65. This is because of thinning operations that remove trees to manage stand density in the intensive regime. The trees that are left take some time to benefit from the more favourable conditions. It is only at a 3% discount rate, using the real options approach, that all three regimes show positive net present value, although the extensive regime still gives the largest return.

In Table 5 we show a single amount for land value that is unrelated to price. This contrasts with stands of age 45 and above shown, in Figure 2, for which value increases with the current price. No matter what the price is at the beginning of a rotation, by the time the stand achieves harvestable volumes, we would expect the price to have reverted toward the long run mean. Thus land values at the beginning of a rotation are insensitive to today's product prices given the parameters we have chosen for our mean reverting process. If we had assumed that price follows a process of geometric Brownian motion, then the value of the bare land would be dependent on today's price.

The values reported in Table 5 would not be consistent with a private firm's perspective on the economics of IFM. For a firm licensed to harvest public lands, a main concern is access to the existing inventory of mature timber over the license period. For the firm the marginal benefits of an investment in IFM in a particular stand are not the extra revenues received from harvesting that stand once it reaches maturity some 50 years hence. Rather, the benefits are more likely perceived to be the added revenues that will result if the regulator permits an immediate increase in the firm's allowable cut of the mature timber inventory [21]. If an investment in intensive forest management today gives the firm greater access to the existing mature inventory, then the investment may well make sense for the firm.

[21] An immediate increase in permitted harvesting levels today in recognition of expected future growth due to investment in IFM has been termed the Allowable Cut Effect (ACE) in the forest industry. Luckert and Haley [35] and Hegan and Luckert [18] provide an analysis of ACE as a policy tool.

Suppose a firm is told by a regulator that it will be given the right to harvest a mature stand of trees, if it agrees to begin a program of IFM once the stand is harvested. The value of this investment opportunity is reflected in the value of a 50 year old stand, shown in the third panel of Figure 2 for the intensive level of management. The values shown here are significantly larger than the values at the beginning of the rotation, reported in Table 5. At a price of zero, V for the intensive regime is about \$7600. It would be worthwhile for the firm to undertake intensive management, if as a result it is given the right to harvest the existing inventory at some time in the future. The firm would, however, make the largest return if it could achieve an increased allowable cut without having to make the investment in intensive management. The economics for the firm would also depend on stumpage and taxes paid to government, neither of which has been included in this analysis.

Although an investment in IFM may make sense for the firm, it is clearly the role of the regulator to compare the marginal costs and benefits of the scheme from society's point of view before embarking on a scheme to encourage firms to make such investments.

5.2 Accuracy of Results

As with any numerical solution methodology we must be concerned with truncation error in the discretizations of time, quantity, and price (Tavella and Randall [51]). The finer the grid used in the solution the more accurate will be our estimated value for V. In Section 10 we show that the grid size that we have chosen to compute the results of Table 5 gives us results to an acceptable degree of accuracy. We also show that we can get a very large change in our answer when we use a grid size that is much coarser than that used to compute our base case results. This is an important point when comparing with the MDP models. We demonstrate in Section 9 that in theory the MDP model and the numerical solution of the LCP are equivalent. However numerical solution of the LCP using a finite difference approach is more efficient than solving the MCP model and hence permits use of a finer grid scheme. In an effort to handle more stochastic variables and keep the solution tractable, MDP models are sometimes solved with a very coarse grid. This raises the question of whether the results are reliable.

5.3 Harvesting Restrictions

As noted previously, there is a long standing tradition for licensees harvesting Ontario public forests to strive to maintain an even flow of wood to mills, which limits a firm's ability to respond to price volatility. Also the maximum annual harvest level is set by government regulation. Wood flow is determined over an entire forest unit consisting of many individual stands. With our stand level model we cannot fully describe the impact of harvesting restrictions, however we can mimic their impact to some extent. In particular we consider the effect of minimum harvesting requirements, which would force a firm to harvest a certain quantity of wood even in times of depressed markets. In the stand level model we mimic this restriction by requiring harvesting at a particular stand age.

Figure 4 shows the value of the option to harvest under extensive management when the restriction is imposed that harvesting must occur when the stand is between the ages of

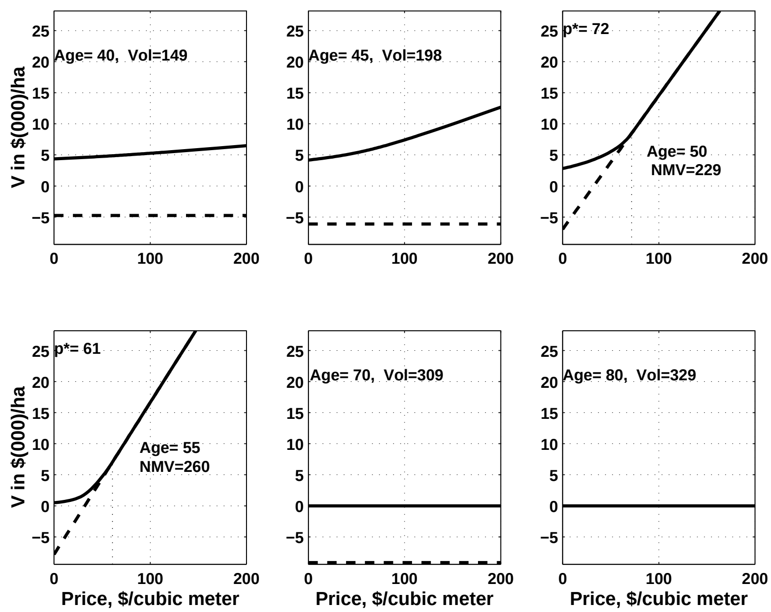


FIGURE 4: *Value of the opportunity to harvest when harvesting must occur between ages 50 and 55, intensive regime, 3% discount rate. Solid line: value of the option to harvest. Dashed line: Payout from harvesting immediately. P^*: critical price at which harvesting is worthwhile.*

50 and 55. If harvesting does not occur within that period it is assumed the firm loses its rights to harvest. Comparing Figure 4 with Figure 2, we note that in the restricted case, value is now fairly insensitive to price at age 40, since it is still 10 years before harvesting can occur. We also note that at ages 50 and 55 the critical prices are lower than for the unrestricted case. By age 55 the critical price has dropped to $61 (compared to $79 for the unrestricted case), reflecting the fact that if harvesting does not occur in that year the land will be worthless to the firm as it will have to give up its license. Under this restriction, the value of the land at the beginning of the rotation is $16/ha, significantly lower than in the unrestricted case of $787/ha.

This exercise demonstrates that the pursuit of an even flow of timber can significantly change the economics of commercial forestry due to their impact on the ability of the forest manager to respond to price volatility. In fact it is only by considering stochastic prices that the costs of these types of restrictions can be fully appreciated.

6 Concluding Comments

This paper has presented a two-factor multi-rotation model of the tree harvesting decision. A contribution is in specifying the problem as a linear complementarity problem (LCP) which is solved using a fully implicit finite difference approach - an approach that is commonly used in the finance literature for valuing real and financial options. We contrast our methodology with other approaches used in the forestry literature to handle optimal rotation with stochastic prices, such as Markov Decision Process (MDP) models. We note that the LCP and the MDP model are in theory equivalent. An important advantage of the LCP approach is that we are assured that the solution will converge to the correct answer (based on a large numerical analysis literature), and we can check easily accuracy by solving for successively finer grids. We demonstrated that the value of the harvesting option varies widely when a coarser solution grid is used, as is sometime done with MDP models in order to handle more state variables.

We used our model to address some policy issues in the Ontario boreal forest. This paper has shown that the value of an investment in forestry can be significantly affected when a firm's ability to react to volatile prices is constrained. Constraints may be due to government regulations, such as allowable cut requirements, or may reflect the structural realities of an industry in which a firms having invested in mill capacity want to maintain a reasonable capacity utilization. There are no doubt costs to a mill if input is highly variable, but these costs should be balanced with the benefits of being able to react optimally to price swings. The value of the option to harvest a stand of trees should be an important consideration in any review of forest management regulations, with the goal of designing regulations that continue to meet environmental constraints, but offer firms the maximum flexibility to manage license areas in the face of price risk.

This paper has also estimated the value of an investment in a stand of trees under three different levels of forest management. Given the assumptions for price, cost, and yield used here, an investment in IFM cannot be justified from a public point of view in terms of the revenues generated by commercial forestry alone. If a firm is given an increase in its allowable cut on the condition that it undertakes IFM, it may well make sense for that firm to undertake the investment. For government regulators the question that remains is what are the benefits, in addition to lumber revenues, that would justify encouraging such an investment by the private sector.

A direction for future research is to extend the methodologies from the finance literature to consider harvesting and other constraints at a forest unit level under stochastic prices. This would involve a multi-stand approach, as well as consideration of incremental mill costs with swings in capacity utilization. Biological and catastrophic risk are other avenues of research using a real options approach.

7 Appendix 1: Numerical Solution of the Linear Complementarity Problem

7.1 Method of characteristics

The solution of Equation (3.9) is accomplished by discretizing the term $\phi V_Q - V_\tau$ in HV by the method of characteristics (Morton and Mayers [38]).

Consider some function $U(X, \tau)$. τ refers to time to expiry of the option, or $T - t$. Then we can write

$$\frac{dU}{d\tau} = U_X X_\tau + U_\tau \tag{7.1}$$

If U satisfies the equation

$$U_\tau + a(X, \tau) U_X = 0 \tag{7.2}$$

then, from Equation (7.1), if we let $a(X,\tau) = X_\tau$, then $dU = 0$ along the characteristic curves defined by

$$\frac{dX}{d\tau} = a(X,\tau). \tag{7.3}$$

If we consider the simple case where $a(X,\tau) = \textit{constant} = \hat{a}$, then the solution to Equation (7.2) is

$$U(X,\tau) = U(X - \hat{a}\tau, 0). \tag{7.4}$$

This can be verified by taking the total derivative of $U(X - \hat{a}\tau, \tau)$ and observing that $dU = 0$ when $X_\tau = \hat{a}$ and $\tau = 0$. In the case when $a(X,\tau) \neq \textit{constant}$, we can still approximate Equation (7.4) in discrete time by

$$\frac{U(X_i,\tau^{n+1}) - U(X_i - a(X_i,\tau^{n+1})\Delta\tau, \tau^n)}{\Delta\tau} + O(\Delta\tau) = 0. \tag{7.5}$$

The basic PDE in the continuation regions for the tree harvesting problem, Equation (3.7), can be written in terms of τ as follows:

$$V_\tau - \phi V_Q = \frac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P)V_P - \rho V + A \tag{7.6}$$

where the left hand side is a function of Q and τ for a fixed P, and the right hand side is a function of P for a fixed Q and τ.

The left hand side of Equation (7.6) looks like the left hand side of Equation (7.2), if $a(X,\Delta\tau) = -\phi$, recalling that $\phi \equiv dQ/dt$, and replacing X with Q and U with V. As will be shown below, this observation allows us to approximate the two factor problem by a solving a series of one dimensional PDE's and employing an interpolation operation at each time step to exchange information between the one dimensional PDE's.

We now consider the numerical solution of the LCP, Equation (3.9), using the characteristic approach and a fully implicit differencing scheme. Define nodes on the axes for P, Q, and τ by

$$\begin{aligned} P &= [P_1, P_2, ..., P_i, ..., P_I] \\ Q &= [Q_1, Q_2, ..., Q_j, ..., Q_J] \\ \tau &= [\tau_1, \tau_2, ..., \tau_n, ...\tau_N] \end{aligned} \tag{7.7}$$

We concern ourselves with the value of the option to harvest at points defined by the three dimensional grid $(P,Q,\tau) = (P_i, Q_j, \tau_n)$. At any point on the grid the value of the option is $V = V(P_i, Q_j, \tau_n) \equiv V_{ij}^n$.

To impose the conditions of the LCP we define a function $\Pi(V)$ to be a penalty term that will prevent the value of the option V from ever falling below the payout from harvesting immediately, $(P - C)Q + V(\tau, P, 0)$ [22]. The penalty term, Π, equals 0 in the continuation region (i.e. when $HV = 0$ and $V > (P - C)Q + V(\tau, P, 0)$) and $\Pi > 0$ when it is optimal to

[22] See Zvan *et al* [62] for a discussion of the penalty method.

harvest (i.e. when $HV > 0$ and $V = (P - C)Q + V(\tau, P, 0)$). If we include the penalty term in Equation (7.6), Equation (3.9), the LCP, can be approximated by

$$V_\tau - \phi V_Q = \frac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P)V_P - \rho V + A + \Pi(V) \tag{7.8}$$

Using Equation (7.5), our difference scheme for Equation 7.8 can be written as

$$\frac{V(P_i, Q_j, \tau^{n+1}) - V(P_i, Q_j + \phi_j \Delta\tau, \tau^n)}{\Delta\tau} = \left[\frac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P)V_P - \rho V + A + \Pi(V)\right]_{ij}^{n+1}. \tag{7.9}$$

Let

$$V^*(P_i, Q_j)^n \equiv V(P_i, Q_j + \phi_j \Delta\tau, \tau_n). \tag{7.10}$$

Then Equation 7.9 can be written as[23]

$$\frac{V(P_i, Q_j)^{n+1} - V^*(P_i, Q_j)^n}{\Delta\tau} = \left[\frac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P)V_P - \rho V + A + \Pi(V)\right]_{ij}^{n+1}. \tag{7.11}$$

Note that the right hand side of Equation (7.11) has derivatives with respect to P only. Therefore Equation (7.11) defines a set of discretized one dimensional partial differential equations, one for each Q_j. These can be solved independently within each time step. After a time step is completed, these one-dimensional PDE's exchange information through the interpolation operation:

$$V^*(P_i, Q_j)^n = V(P_i, Q_j + \phi_j \Delta\tau, \tau_n). \tag{7.12}$$

We used linear interpolation.

Within each time step the LCP, Equation (3.9), is solved for fixed Q_j using a fully implicit finite difference method and the penalty method, as described in Insley [25]. The right hand side of Equation (7.11) is discretized using a central , forward or backward difference scheme as appropriate, as in Insley [25].

7.2 Discretization for $i = [2, ..., I - 1]$

We now concern ourselves with the discretization of the RHS of Equation (7.11) and make use of the following definitions:

$$\alpha_i \equiv \frac{\sigma^2 P_i^2}{\Delta P_{i+1/2} + \Delta P_{i-1/2}}; \qquad \beta_i \equiv \frac{\eta(\bar{P} - P_i)}{\Delta P_{i+1/2} + \Delta P_{i-1/2}} \tag{7.13}$$

where $\Delta P_{i+1/2} \equiv P_{i+1} - P_i$ and $\Delta P_{i-1/2} \equiv P_i - P_{i-1}$. Using a fully implicit approach and central differencing along the P axis, our difference scheme when $i = 2, ..., I - 1$ for P_i and $j = 1, ..., J$ for Q_j is:

[23] See Bermejo [3].

$$\frac{V(P_i,Q_j,\tau^{n+1}) - V(P_i,Q_j+\phi_i\Delta\tau,\tau^n)}{\Delta\tau} = \left\{\frac{\sigma^2 P^2}{2}\left[\frac{\frac{V_{i+1,j}-V_{ij}}{\Delta P_{i+1/2}} - \frac{V_{ij}-V_{i-1,j}}{\Delta P_{i-1/2}}}{\frac{\Delta P_{i+1/2}-\Delta P_{i-1/2}}{2}}\right]\right. \tag{7.14}$$

$$\left. +\eta(\bar{P}-P)\left[\frac{V_{i+1,j}-V_{i-1,j}}{\Delta P_{i+1/2}+\Delta P_{i-1/2}}\right] - \rho V_{ij} + A + \frac{\pi_{ij}}{\Delta\tau}[(P_i - C)Q_j + V_{i0} - V_{ij}]\right\}^{n+1}.$$

The superscript $n+1$ on the right hand side means that all variables within the curly brackets are evaluated at $\tau = n+1$.

Extensive manipulation of the right hand side of Equation (7.14) results in:

$$RHS \equiv \Big[a_i V_{i-1,j} + b_i V_{i+1,j} - (a_i + b_i + \delta)V_{ij} + A \tag{7.15}$$
$$+\frac{\pi_{ij}}{\Delta\tau}[(P_i - C)Q_j + V_{i0} - V_{ij}]\Big]^{n+1}$$

where

$$a_i \equiv \frac{\alpha_i}{\Delta P_{i-1/2}} - \beta_i; \qquad b_i \equiv \frac{\alpha_i}{\Delta P_{i+1/2}} + \beta_i \tag{7.16}$$

The last term on the right hand side of Equation (7.15) is the discretized penalty function $\Pi(V)$, where

$$\pi_{ij}^{n+1} = 0 \text{ if } V_{ij}^{n+1} \geq (P_i - C)Q_j + V_{i0} \tag{7.17}$$
$$\pi_{ij}^{n+1} = L \text{ if } V_{ij}^{n+1} < (P_i - C)Q_j + V_{i0}$$

where L is a suitably large number. In this paper L is set to 10^6.

To avoid an oscillatory solution it is necessary that a and b from Equation 7.16 both be non-negative. Since β can be either positive or negative, a and b can be either positive or negative, but if one is positive the other is negative. If $a < 0$ a forward difference scheme for P is used to avoid oscillations, rather than the central scheme shown in Equation 7.14. If b is negative then a backward difference scheme must be used for P (Zvan *et al* [62]).

Using a forward difference scheme a and b in Equation 7.15 are defined as

$$a_i \equiv \frac{\alpha_i}{\Delta P_{i-1/2}}; \qquad b_i \equiv \frac{\alpha_i}{\Delta P_{i+1/2}} + \gamma_i, \text{ where } \gamma_i = \frac{\eta(\bar{P}-P)}{\Delta P_{i+1/2}} \tag{7.18}$$

Using a backward difference scheme a and b in Equation (7.15) are defined as

$$a_i \equiv \frac{\alpha_i}{\Delta P_{i-1/2}} - \theta_i; \qquad b_i \equiv \frac{\alpha_i}{\Delta P_{i+1/2}} \text{ where } \theta_i = \frac{\eta(\bar{P}-P)}{\Delta P_{i-1/2}} \tag{7.19}$$

7.3 Discretization of Boundary Conditions for P

When $i = 1$ no special boundary condition is required for P, and a forward discretization scheme can be used. As $P \to 0$, Equation (7.11) becomes

$$\frac{V(P_i,Q_j)^{n+1} - V^*(P_i,Q_j)^n}{\Delta\tau} = \Big[\eta\bar{P}V_P - \rho V + A + \Pi(V)\Big]^{n+1}_{i=1,j}. \tag{7.20}$$

A forward discretization is then

$$\frac{V(P_i,Q_j)^{n+1}-V^*(P_i,Q_j)^n}{\Delta\tau} = \left[b_iV_{i+1}-(b_i+\rho)V_i+A+\frac{\pi_{ij}}{\Delta\tau}[(P_i-C)Q_j+V_{i0}-V_{ij}]\right]^{n+1}_{1,j} \quad (7.21)$$

with b defined as in Equation (7.18). In Equation (7.18) as $P\to 0$ we see that $\alpha=0$ and hence $a=0$ and also, $b=\gamma$.

When $i=I$ we set $V_{PP}=0$ in Equation (7.11) and use a backward difference scheme which gives:

$$\frac{V(P_i,Q_j)^{n+1}-V^*(P_i,Q_j)^n}{\Delta\tau} = \left[\frac{\eta(\bar{P}-P)}{\Delta P_{i-1/2}}V_{i-1}+(\frac{\eta(\bar{P}-P)}{\Delta P_{i-1/2}}-\rho)V_i+A+\frac{\pi_{ij}}{\Delta\tau}[(P_i-C)Q_j+V_{i0}-V_{ij}]\right]^{n+1}_{I,j}. \quad (7.22)$$

7.4 Iterative Solution

Equations (7.14), (7.17), (7.21), and (7.22) represent a system of nonlinear algebraic equations, due to the penalty term. Hence, Equation (7.17), and must be solved iteratively. A description of the iterative solution for a single rotation problem is given in Insley [25]. The iterative solution can best be understood if we write the system of equations in matrix form. To this end, define the vectors

$$V^{n+1}\equiv\begin{bmatrix}V_1^{n+1}\\V_1^{n+1}\\\vdots\\V_I^{n+1}\end{bmatrix};\quad V^{*n}\equiv\begin{bmatrix}V_1^{*n}\\V_1^{*n}\\\vdots\\V_I^{*n}\end{bmatrix};\quad P^n\equiv\begin{bmatrix}P_1^n\\P_1^n\\\vdots\\P_I^n\end{bmatrix};\quad A^n\equiv\begin{bmatrix}A_1^n\\A_1^n\\\vdots\\A_I^n\end{bmatrix}; \quad (7.23)$$

Also, define a diagonal matrix as:

$$\begin{aligned}\bar{Q}(V_{ii}^{n+1}) &= L \text{ if } V^{n+}<(P_i^{n+1}-C)Q_j-V_{i0}^{n+1} \quad (7.24)\\ &= 0 \text{ otherwise}\\ \bar{Q}(V_{ij}^{n+1}) &= 0 \text{ if } i\neq j\end{aligned}$$

Let B be an $I\times I$ matrix. Multiplying B by the vector V^{n+1} gives a vector BV^{n+1} with the following elements:

$$\begin{bmatrix}\left[\frac{\Delta\tau\eta\bar{P}}{\Delta P_{i+1/2}}+\rho\right]V_1^{n+1}-\left[\frac{\Delta\tau\eta\bar{P}}{\Delta P_{i+1/2}}\right]V_2^{n+1}\\ -\Delta\tau b_2V_1^{n+1}+\Delta\tau(a_2+b_2+\rho)V_2^{n+1}-\Delta\tau a_2V_3^{n+1}\\ \vdots\\ -\Delta\tau b_iV_{i-1}^{n+1}+\Delta\tau(a_i+b_i+\rho)V_i^{n+1}-\Delta\tau a_iV_{i+1}^{n+1}\\ \vdots\\ -\Delta\tau b_{I-1}V_{I-2}^{n+1}+\Delta\tau(a_{I-1}+b_{I-1}+\rho_{I-1})V_{I-1}^{n+1}-\Delta\tau a_{I-1}V_I^{n+1}\\ \left[\Delta\tau\frac{\eta(\bar{P}-P_I)}{\Delta P_{I-1/2}}\right]V_{I-1}^{n+1}-\left[\Delta\tau\frac{\eta(\bar{P}-P)}{\Delta P_{I-1/2}}-\rho\right]V_I^{n+1}\end{bmatrix} \quad (7.25)$$

Equations (7.14), (7.17), (7.21), and (7.22) can be written as:

$$BV^{n+1} + \left[I + \bar{Q}(V^{n+1})\right] V^{n+1} = V^{*n} + \Delta\tau A^n + \bar{Q}(V^{n+1}) \left[(P_i^{n+1} - C)Q_j - V_{i0}^{n+1}\right] \quad (7.26)$$

where I is the identity matrix. We can express V^{*n} as $V^{*n} = FV^n$ where F is an interpolation matrix. For linear interpolation, F has the properties that its entities are non- negative and all row sums are one. Equation (7.26) is solved iteratively as described in Insley [25].

8 Appendix 2: Markov Decision Process Models and the Linear Complementarity Problem

The key to a Markov decision model is the Markov matrix, or transition probability matrix of the model.[24] Consider a random variable X. We refer to X_n as the outcome of the nth trial. The state space is labelled by nonnegative integers $(0, 1, 2, 3, ...)$. The probability of X_{n+1} being in state j, given that X_n is in state i (a one-step transition probability) is denoted by $P_{ij}^{n,n+1}$, i.e.

$$P_{ij}^{n,n+1} = Pr\{X_{n+1} = j \mid X_n = i\} \quad (8.1)$$

When the one-step transition probabilities are independent of time, the Markov process is said to have stationary transition probabilities. In this case $P_{ij}^{n,n+1} = P_{ij}$. P_{ij} is the probability that the state value goes from state i to j in one trial. All the P_{ij} can be arranged in a matrix referred to as the Markov matrix or the transition probability matrix of the process, denoted by P. An example of P for four different states is:

$$P = \begin{pmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{pmatrix} \quad (8.2)$$

The $(i+1)$st row of P is the probability distribution of the values of X_{n+1} under the condition $X_n = i$. The order of the matrix (number of rows) is equal to the number of states. Given four different states the quantities P_{ij} satisfy:

$$\begin{aligned} P_{ij} \geq & \ 0, \qquad i, j = 0, 1, 2, 3 \quad (8.3) \\ \sum_{j=0}^{3} P_{ij} = & \ 1, \qquad i = 0, 1, 2, 3 \end{aligned}$$

We can show how a matrix with all the properties of the Markov matrix is derived in the solution of the LCP. For convenience we define a variable W such that $V = e^{-\rho\tau}W$. Thus

$$V_\tau = -e^{-\rho\tau}\rho W + e^{-\rho\tau}W_\tau \quad (8.4)$$

[24] Two references that discuss Markov transition probability matrices are Karlin and Taylor [27] and Hillier and Lieberman [19].

Substituting these expressions into Equation (7.8) and setting $A = 0$, we get

$$W_\tau - \phi W_Q = \frac{1}{2}\sigma^2 P^2 W_{PP} + \eta(\bar{P} - P)W_P + e^{\rho\tau}\Pi(e^{-\rho\tau}W) \tag{8.5}$$

We can discretize Equation (8.5) as shown in Sections (8.2) and (8.3). We end up with an equation similar to Equation (7.26). We are ignoring the penalty term here for simplicity.

$$\left[\hat{B} + I\right] W^{n+1} = W^{*n} \tag{8.6}$$

The matrix $[\hat{B}]$ has the following elements:

$$\hat{B} = \begin{bmatrix} \frac{\Delta\tau\eta\bar{P}}{\Delta P_{i+1/2}} & \frac{-\Delta\tau\eta\bar{P}}{\Delta P_{i+1/2}} & 0 & 0 & 0 & ... & 0 & 0 \\ -\Delta\tau b_2 & \Delta\tau(a_2 + b_2) & -\Delta\tau a_2 & 0 & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 & -\Delta\tau b_{I-1} & \Delta\tau(a_{I-1} + b_{I-1}) & -\Delta\tau a_{I-1} \\ 0 & 0 & 0 & ... & 0 & 0 & \frac{\Delta\tau\eta(\bar{P}-P_I)}{\Delta P_{I-1/2}} & \frac{-\Delta\tau\eta(\bar{P}-P_I)}{\Delta P_{I-1/2}} \end{bmatrix} \tag{8.7}$$

It follows that

$$W^{n+1} = \left[\hat{B} + I\right]^{-1} W^{*n} \tag{8.8}$$

We know that $V^{n+1} = e^{-\rho\Delta\tau}W^{n+1}$. It follows that

$$V^{n+1} = e^{-\rho\Delta\tau}\left[\hat{B} + I\right]^{-1} V^{*n} \tag{8.9}$$

Recall that $V^{*n} = FV^n$ where F is the interpolation matrix.

We note that $[\hat{B} + I]$ will have positive diagonals, non-positive off-diagonals, and rows that sum to 1.

Lemma 1: The row sums of $[\hat{B} + I]^{-1}F$ are unity.

Proof: Let e be an $[I \times 1]$ column vector of ones. We can observe that $[\hat{B} + I]Fe = Fe$ (since $\hat{B}e = 0$ and $Fe = e$). This implies that $Fe = [\hat{B} + I]^{-1}Fe$. The lemma follows since

$$\text{rowsum}_i[\hat{B} + I]^{-1}F = \left[[\hat{B} + I]^{-1}Fe\right]_i = [Fe]_i = 1 \tag{8.10}$$

QED

Lemma 2: All elements of $[\hat{B} + I]^{-1}F$ are non-negative.

Proof: $[\hat{B} + I]$ has positive diagonals, non-positive off-diagonals, and is diagonally dominant. It may therefore be classified as an M-matrix, which has the property that all elements of its inverse are non-negative. (Varga [55]) We stated previously that the elements of F are all non-negative. *QED*

Theorem: $[\hat{B} + I]^{-1}F$ satisfies (8.4) and hence is a Markov matrix.

Proof: By lemma 2, all elements of $[\hat{B} + I]^{-1}F$ are non-negative. By lemma 1, the row sums of $[\hat{B} + I]^{-1}F$ are all unity. Hence all elements must be less than or equal to 1. *QED*

We can see that $[\hat{B}+I]^{-1}F$ is equivalent to a Markov transition matrix showing the probability of moving from one state at τ_n to another at τ_{n+1}. From Equation (8.9), the value of the option at period $n+1$ is equal to the value in period n multiplied by the Markov transition matrix and a discount factor.

Solving the Markov chain decision model is an alternative method to solving the partial differential equation, Equation (7.8). The Markov matrix is frequently estimated through simulation. This is unnecessary with the approach used in this paper. Instead we discretize the PDE and solve the resulting system of equations iteratively using well established numerical techniques. There is no need to directly estimate the Markov matrix. Note that the elements of $(B+I)^{-1}F$ are to $O(\Delta\tau^2)$ equal to the discretized Green's function that must solve the PDE.

$$V_\tau - \phi V_Q = \frac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P}-P)V_P - \rho V + A + \delta(P-P')\delta(\tau-\tau'). \tag{8.11}$$

where $\delta(\tau-\tau')$ represents the delta function (see Wilmott [58]). The Green's function of this PDE is simply the transition probability density function, $P(P,t,P',t')$ relating the probability of a transition from state $(P,t)\rightarrow(P',t')$. This is also the solution of the forward Kolmogorov equation.

Although the Green's function (hence the transition density function) can be determined in some simple cases, it is not in general possible to obtain an analytic solution. However we can always discretize the PDE directly and hence obtain an approximate transition matrix to $O(\Delta t^2)$. Note that since we solve the linear complimentarity problem at each timestep we are directly enforcing the optimal control, which ensures that the solution satisfies the smooth pasting condition. The usual stochastic dynamic programming approach applies a control in explicit fashion, hence the solution is in an inconsistent state after each time step. This will be an issue only in non-autonomous problems, but will not matter when we are solving for a steady state, as in this paper.

Note that we do not compute and solve $(\hat{B}+I)^{-1}$, but solve $(\hat{B}+I)V^{n+1}=V^{*n}$ which is considerably more efficient, since $(B+I)$ is tridiagonal.

9 Appendix 3: Convergence of the Finite Difference Scheme

The accuracy of the numerical solution to the PDE depends critically on the number of nodes used for each factor: price (P), volume (Q), and time (τ).[25] The more nodes, the more accurate the solution, but the longer the solution time. An upper limit on the number of nodes will eventually be reached based on the memory limits of the computer. It is important to check whether an acceptable number of nodes has been included by noting how much the solution changes as the grid is refined. Specifically we observe the change in the answer as we double the total number of nodes.

[25] Wilmott [57] and Tavella [51] discuss many of the issues referred to in this appendix. The Greens function and backward Kolmogorov equation are also discussed in Karlin and Taylor [28] and Dixit and Pindyck [11].

Grid	# of Q nodes	# of P nodes	time step size (years)	V ($/ha)	Change in V	Ratio
Coarse	54	37	0.25	1641.2		
Medium	107	73	0.125	1652.0	10.8	
Fine	0	0	0.0625	1658.4	6.0	1.7

TABLE 6: *Convergence of solution as grid is refined for the basic regime, 3% discount rate*

	Very coarse grid	Medium grid
Price $\$/m^3$ $P = [0...277]$	20 nodes	73 nodes
Quantity $\$/ha$ $Q = [0...357]$	25 nodes	107 nodes
time step size	0.25 years	0.125 years
Value at $t = 0$		
extensive	$962	$ 1660
basic	-$489	$ 1652
intensive	-$553	$ 787

TABLE 7: *Comparing solutions for very coarse grid and medium grid. Value of the land at the beginning of the first rotation, $/hectare, 3% discount rate*

More formally, let $\Delta\tau = c_1 h$, $\Delta P = c_2 h$, $\Delta Q = c_3 h$. Since first order timestepping is being used, and the cumulative effect of linear interpolation in the Q direction will be $O(\frac{\Delta Q^2}{\Delta \tau}) = O(h)$, we expect that the computed solution V^{comp} is related to the exact solution V^{exact} by

$$V_h^{comp} = V^{exact} + c_4 h, \; h \to 0. \tag{9.1}$$

This implies that

$$\frac{V_h - V_{h/2}}{V_{h/2} - V_{h/4}} \sim 2, \; h \to 0. \tag{9.2}$$

In Table 6 we see that for the basic regime this ratio is about 1.7, indicating that we are near the asymptotic convergence range. In particular, we can be confident that the solution on the finest grid is 1658 ± 6 or accurate to within about 0.4%.

The importance of testing for truncation error becomes evident when we observe how our answer changes when we significantly reduce the number of nodes. This is shown in Table 7 for the three management intensities. We compare the medium grid which was used to calculate the real options values in Table 5 with a very coarse grid. We observe a huge change in our estimated V values as we move from the medium to the very coarse grid.

References

[1] L. Anderson. A simple approach to the pricing of Bermudan swaptions in the multifactor libor market model. *Journal of Computational Finance*, 3(2):1–32, 2000.

[2] Kenneth J. Arrow and Anthony C. Fisher. Environmental preservation, uncertainty, and irreversibility. *Quarterly Journal of Economics*, LXXXVIII:312–319, 1974.

[3] R. Bermejo. A galerkin characteristic transport-difusion equation. *SIAM Journal of Numerical Analysis*, 32:425–454, 1995.

[4] Ontario Forest Accord Advisory Board. Room to Grow, Final Report of the Ontario Forest Accord Advisory Board, March 2002. Available from the Ontario Government web site `http://www.ontarioslivinglegacy.com/spectrasites/internet/oll/ofaab.cfm`.

[5] Richard J. Brazee, Gregory S. Amacher, and M. Christine Conway. Optimal harvesting with autocorrelated stumpage prices. *Journal of Forest Economics*, 5(2):201–216, 1999.

[6] R.J. Brazee and D.H. Newman. Observations on recent forest economics research on risk and uncertainty. *Journal of Forest Economics*, 5(2):193–200, 1999.

[7] W.A Brock and M. Rothschild. Comparative statics for multidimensional optimal stopping problems. In H. Sonnenschein, editor, *Models in Economic Dynamics, Lecture Notes in Economics and Mathematical Systems Series, vol.264*, pages 124–138. Springer-Verlag, 1986.

[8] W.A Brock, M. Rothschild, and J.E. Stiglitz. Stochastic capital theory. In G. Feiwal, editor, *Essays in Honour of Joan Robinson*. Cambridge University Press, 1988.

[9] Joseph Buongiorno. Generalization of faustmann's formula for stochastic forest growth and prices with markov decision process models. *Forest Science*, 47(4):466–474, 2001.

[10] H.R. Clarke and W.J. Reed. The tree-cutting problem in a stochastic environment. *Journal of Economic Dynamics and Control*, 13:569–595, 1989.

[11] A.K. Dixit and R.S. Pindyck. *Investment Under Uncertainty*. Princeton University Press, 1994.

[12] A.K. Dixit, R.S. Pindyck, and Sigbjorn Sødal. A markup interpretation of optimal investment rules. *The Economic Journal*, 109:179–189, 1999.

[13] C.M. Elliott and J.R. Ockendon. *Weak and Variational Methods for Free and Moving Boundary Problems*. Pitman, 1982.

[14] A. Friedman. *Variational Principles and Free Boundary Problems*. Robert Krieger Publishing, 1988.

[15] Peichen Gong. Optimal harvest policy with first-order autoregressive price process. *Journal of Forest Economics*, 5(3):413–439, 1999.

[16] R. G. Haight and T. P. Holmes. Stochastic price models and optimal tree cutting: Results for loblolly pine. *Natural Resource Modeling*, 5:423–443, 1991.

[17] James D. Hamilton. *Time Series Analysis*. Princeton University Press, 1994.

[18] R. L. Hegan and M. K. Luckert. An economic assessment of using the allowable cut effect for enhanced forest management policies: an Alberta case study. *Canadian Journal of Forest Research*, 30:1591–1600, 2000.

[19] F.S. Hillier and G.J. Lieberman. *Introduction to operations research*. McGraw-Hill, 1990.

[20] James N. Hool. A dynamic programming-Markov chain approach to forest production control. *Forest Science Monograph*, 12:1–26, 1966.

[21] John C. Hull. *Options, futures, and other derivatives*. Prentice Hall, 2003.

[22] L. Hultkrantz. The behaviour of timber rents in Sweden 1909-1990. *Journal of Forest Economics*, 1(2):165–180, 1995.

[23] A-J Hunnemeyer. *Private Resource Management and Public Trust: Optimal Design of Forest Conservation Contracts in Ontario*. PhD thesis, University of Guelph, Guelph, Ontario, February 2001.

[24] A-J Hunnemeyer and Kimberly Rollins. Private use and public trust. Presented at the World Congress of Environmental and Resource Economists, Monteray, California, June 2002.

[25] M. Insley. A real options approach to the valuation of a forestry investment. *Journal of Environmental Economics and Management*, 44:471–492, 2002.

[26] C. Kao. Optimal stocking levels and rotation under risk. *Forest Science*, 28:711–719, 1982.

[27] Samuel Karlin and Howard M. Taylor. *A first course in stochastic processes*. Academic Press, 1975.

[28] Samuel Karlin and Howard M. Taylor. *A second course in stochastic processes*. Academic Press, 1981.

[29] Ismail Kaya and Joseph Buongiorno. Economic harvesting of uneven-aged northern hardwood stands under risk: a Markovian decision model. *Forest Science*, 33:889–907, 1987.

[30] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and their Applications*. Academic Press, 1980.

[31] M. R. Lembersky and K. Norman Johnson. Optimal policies for managed stands: an infinite horizon Markov decision process approach. *Forest Science*, 21:109–122, 1975.

[32] Ching-Rong Lin and Joseph Buongiorno. Tree diversity, landscape diversity, and economics of maple-birch forests: implications of Markovian models. *Management Science*, 44:1351–1366, 1998.

[33] P. Lohmander. Pulse extraction under risk and a numerical forestry example. *Syst. Anal. Model. Simulation*, 4:339–354, 1988.

[34] F.A. Longstaff and E.S. Schwartz. Valuing american options by simulation: A least squares approach. *Review of Financial Studies*, 14:113–47, 2001.

[35] M. K. Luckert and R. L. Hegan. The allowable cut effect as a policy instrument in Canadian forestry. *Canadian Journal of Forest Research*, 25:1821–1829, 1995.

[36] R.A. Miller and K. Voltaire. A stochastic analysis of the tree paradigm. *Journal of Economic Dynamics and Control*, 6:371–386, 1983.

[37] R. Morck, E. Schwartz, and D. Strangeland. The valuation of forestry resources under stochastic prices and inventories. *Journal of Financial and Quantitative Analysis*, 4:473–487, 1989.

[38] K. W. Morton and D. F. Mayers. *Numerical Solution of Partial Differential Equations*. Cambridge University Press, 1994.

[39] David H. Newman. Forestry's golden rule and the development of the optimal rotation literature. *Journal of Forest Economics*, 8:7–27, 1998.

[40] Carl J. Norstrom. A stochastic model for the growth period decision in forestry. *Swedish Journal of Economics*, 77:329–337, 1975.

[41] Ontario Ministry of Natural Resources. 1999 Ontario Forest Accord, March 1999. Available at `http://www.mnr.gov.on.ca/mnr/oll/ofaab/accord.html`.

[42] Ontario Ministry of Natural Resources. Annual Report on Forest Management. September 2001.

[43] A. J. Plantinga. The optimal timber rotation: An option value approach. *Forest Science*, 44:192–202, 1998.

[44] W. Reed and H. Clarke. Harvest decisions and asset valuation for biological resources exhibiting size-dependent stochastic growth. *International Economic Review*, 31:147–169, 1990.

[45] Resource Information Systems, Inc. Ontario's private timber market. Study commissioned by the Ontario Ministry of Natural Resources, May 2001.

[46] Sigbjørn Sø dal. The stochastic rotation problem, a comment. *Journal of Economic Dynamics and Control*, 26:509–515, 2002.

[47] K. Rollins, M. Forsyth, S. Bonti-Ankomah, and B. Amoah. A financial analysis of a white pine improvement cut in Ontario. *The Forestry Chronicle*, 71:466–471, 1995.

[48] J.-D. Saphores, L. Khalaf, and D. Pelletier. On jump and ARCH effects in natural resource prices: An application to Pacific Northwest stumpage prices. *American Journal of Agricultural Economics*, 85, May 2002.

[49] E. S. Schwartz. The stochastic behaviour of commodity prices: Implications for valuation and hedging. *Journal of Finance*, 52:923–973, 1997.

[50] Domingo A. Tavella. *Quantitative Methods in Derivatives Pricing.* John Wiley and Sons, 2002.

[51] Domingo A. Tavella and Curt Randall. *Pricing Financial Instruments, The Finite Difference Method.* John Wiley and Sons, 2000.

[52] Lawrence D. Teeter and Jon P. Caulfield. Stand density management strategies under risk: effects of stochastic prices. *Canadian Journal of Forest Research*, 21:1373–79, Fall 1991.

[53] T.A. Thomson. Optimal forest rotation when stumpage prices follow a diffusion process. *Land Economics*, 68:329–342, 1992.

[54] L. Trigeorgis. *Real Options, Managerial Flexibility and Strategy in Resource Allocations.* MIT Press, 1996.

[55] Richard S. Varga. *Matrix Iterative Analysis.* Springer Verlag, 2000.

[56] Yngve Willassen. The stochastic rotation problem: a generalization of faustmann's formula to stochastic forest growth. *Journal of Economic Dynamics and Control*, 22:573–596, 1998.

[57] P. Wilmott. *Derivatives, The Theory and Practice of Financial Engineering.* John Wiley & Sons, 1998.

[58] P. Wilmott, J. Dewynne, and S. Howison. *Option Pricing, Mathematical Models and Computation.* Oxford Financial Press, 1993.

[59] R. Yin and D. Newman. A note on the tree cutting problem in the stochastic environment. *Journal of Forest Economics*, 1:181–190, 1995.

[60] R. Yin and D. Newman. Are markets for stumpage informationally efficient? *Canadian Journal of Forest Research*, 26:1032–1039, 1996.

[61] R. Yin and D. Newman. When to cut a stand of trees. *Natural Resource Modeling*, 10:251–261, 1997.

[62] R. Zvan, P.A. Forsyth, and K.R. Vetzal. Penalty methods for American options with stochastic volatility. *Journal of Computational and Applied Mathematics*, 9:199–218, 1998.